MARYAM SAMAVAKI

This thesis presents some new solutions based on linearized the homogeneous Navier-Stokes equations on Riemannian manifolds, related to the Lie bracket of Killing vector fields. The equations for the Killing and conformal Killing vector fields which are overdetermined systems of PDE can be formulated as an eigenvalue problem. Moreover, analyzing several classes of Riemannian manifolds can be interpreted as overdetermined PDE systems whose unknowns are the Riemannian metric components.
Maryam Samavaki

NAVIER–STOKES EQUATIONS ON RIEMANNIAN MANIFOLDS

\[
\begin{align*}
  u_t + \nabla_u \cdot \mu Lu + \nabla(p) &= f \\
  \nabla(u) &= 0
\end{align*}
\]

ACADEMIC DISSERTATION

To be presented by the permission of the Faculty of Science and Forestry for public examination through video connection at the University of Eastern Finland, Joensuu, on May 29, 2020, at 12 o’clock noon. Audience can join the video connection by using the link https://stream.lifesizecloud.com/extension/3262074/c955c636-c1e9-46ad-968b-2a7510b57332

If the case of problems, email maryam.samavaki@uef.fi

University of Eastern Finland
Department of Physics and Mathematics
Joensuu 2020
Author’s address: University of Eastern Finland
Department of Physics and Mathematics
P.O. Box 111
FI-80101 Joensuu
FINLAND
email: maryam.samavaki@uef.fi

Supervisors: Professor Jukka Tuomela
University of Eastern Finland
Department of Physics and Mathematics
P.O. Box 111
FI-80101 Joensuu
FINLAND
email: jukka.tuomela@uef.fi

Reviewers: Professor Elizabeth Mansfield
University of Kent
School of Mathematics, Statistics and Actuarial Science
Canterbury
CT2 7NZ Kent
ENGLAND
email: e.l.mansfield@kent.ac.uk
Professor Jouni Parkkonen
University of Jyväskylä
Department of Mathematics and Statistics
P.O. Box 35
40014 Jyväskylä
FINLAND
email: jouni.t.parkkonen@jyu.fi

Opponent: Professor Mikko Salo
University of Jyväskylä
Department of Mathematics and Statistics
P.O. Box 35
40014 Jyväskylä
FINLAND
email: mikko.j.salo@jyu.fi
ABSTRACT

This thesis presents some new solutions regarding the homogeneous Navier-Stokes equations on Riemannian manifolds. One can write this system on the Riemannian manifold $M$ as

$$u_t + \nabla u \cdot u - \mu Lu + \text{grad}(p) = 0,$$
$$\text{div}(u) = 0,$$

where $u$ is the velocity of the fluid, $p$ is fluid pressure and $L$ is considered a diffusion operator. Through linearizing, we provide many new results and then present more solutions with the bracket of Killing fields for linearized Navier-Stokes equations.

The equations identified for the Killing and conformal Killing vector fields are overdetermined PDE systems and difficult to resolve numerically. We therefore try to find a solution to them by the eigenvalue problem, which is then solved using finite element techniques.

Many classes of Riemannian manifolds are considered, described by certain condition cases imposed by the Ricci tensor, for instance: Ricci recurrent, Cotton, quasi Einstein and pseudo Ricci symmetric conditions. In this context, we try to find possible solutions for overdetermined PDE systems whose unknown components are the Riemannian metric, and, perhaps, also other auxiliary functions.

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Joensuu, January 20, 2020

Maryam Samavaki
LIST OF PUBLICATIONS

This thesis is organised in the following way. First, an introduction to the subject is presented, together with a summary of the papers that this thesis is based on. Secondly, the papers are given in full text, and are:


AUTHOR’S CONTRIBUTION

The publications selected in this dissertation are original research papers. For paper I, first, the second author developed the fundamental idea of the proposed method. Second, both authors made an equal contribution to elaborate on the method. Paper II is based on several formulas, which were computed in Paper I. The main contribution of the second author was in the theoretical background and planning and implementation of the symbolic computation components. The main contribution of the first author was to perform the numerical calculations and generation of vivid figures. Paper III is based on the master’s thesis of the first author published in 2008 [4].
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1 Introduction

There is no doubt that mathematics is the universe’s language, transmitting the signals in the mind to the general relativity equations that qualify the star-galaxy motion. However, almost all people find mathematics boring, dry and irrelevant. Riemannian geometry has evolved into one of the most important aspects in modern mathematics by expanding on 20th century ideas. Thus, the mathematical language of physics has been transformed by Riemannian geometry. Moreover, different physical findings need to be carried out extensively by nonlinear analysis. In this thesis we use differential geometry to study partial differential equations (PDEs).

The chapters of this dissertation introduce and then expand on several topics related to the Navier–Stokes equations, leading to some new contributions to this area of study. Then we apply partial differential equations on Riemannian manifolds. Finally, we concentrate on several classes of the Ricci tensor. These topics can be summarised below.

Initially, we study properties of the Navier–Stokes system on compact Riemannian manifolds, which play a significant role in the domain of fluid dynamics; this system represents a large class of fluid flows. The existence and uniqueness problem of Navier–Stocks equations in three-dimensional space gained its creators a million-dollar prize. The Navier–Stokes system is dominant in numerous natural hydrodynamic phenomena and by solving this system we can obtain parameters such as pressure, velocity, and water level in the desired phenomenon. This system can be written in $\mathbb{R}^n$ as follows:

$$u_t + u \nabla u - \mu \Delta u + \nabla p = f$$
$$\nabla \cdot u = 0.$$

The significant problem in solving such these equations is that there are four equations with unknowns $(u, p)$, where $u$ is the velocity of the fluid and $p$ fluid pressure. The first equation consists of three equations which are all defined in different directions in $\mathbb{R}^3$ with $(u, p)$ as their unknowns. However, the second equation, which is named the mass conservation equation, consists of one equation where velocity is its unknown. For this system, we will present a special solution group that can currently be accessed by the Killing vector fields, because of nonlinearity. Our main results concern the decomposition of the flows to the Killing component and its orthogonal complement. A different conclusion is valid when a diffusion operator is replaced by another operator and the Killing fields by harmonic vector fields. The solutions obtained are therefore completely different, depending on the type of diffusion operator.

In chapter 3, we use FREEFEM++ and Maple software to solve PDEs. FREEFEM++ is a free software platform numerically based on finite element methods. It is also a modeling support in the sense that it helps to obtain easy numerical results that are useful for modifying a physical model, clearing the way for mathematical modeling research. For some computations, Packages Differential Geometry and PDE Tools in Maple have been very useful. Maple is math software that combines the most powerful math application in the world with an interface that makes analyzing,
exploring, visualizing, and solving math problems incredibly easy. We define equations for the Killing and conformal Killing vector fields which are overdetermined systems of PDE. Then we formulate our system as an eigenvalue problem and show that the problem is well posed.

Finally, we analyze several classes of Riemannian manifolds which are described by first imposing certain conditions on the Ricci tensor and then presenting examples of them. These conditions can be interpreted as overdetermined PDE systems whose unknowns are the Riemannian metric components and perhaps also other auxiliary functions.
2 On Navier–Stokes equations

By increasing human knowledge in the fields of mathematics and physics, it was possible, through the use of mathematical equations and physical concepts, to discover and understand many laws of natural phenomena and explain how they can be expressed through mathematical equations. Newton’s three laws are a prime example to substantiate this claim. The flow of various fluids and the properties of each of these fluids are another example and various relationships have been suggested for them. Renowned scientists like Euler, Bernoulli, Navier and Stokes have contributed greatly to understanding the properties of fluids and various currents. Navier–Stokes equations are arguably the most complete set of fluid flow equations to date. The nonlinear system of Navier–Stokes equations forms the mathematical model governing motions, currents, and fluid dynamics (whether liquids or gases).

Section 2.1 addresses the standard way of writing the Navier–Stokes equations in $\mathbb{R}^n$ and different choices for the system is diffusion operator on the Riemannian manifolds, which are necessary for the rest of this chapter. These include the Bochner Laplacian and Hodge Laplacian operators. Section 2.2 introduces some basic ideas for choosing Killing and Harmonic vector fields as a solution to the Navier–Stokes equations. Section 2.3 contains various important integrals that are practical in resolving this system. Two different operators are provided in the next section which can help us write the Navier–Stokes equations in two- and three-dimensional space and then briefly examine different properties of this system in those spaces.

2.1 PERCEPTION-BASED NAVIER–STOKES EQUATIONS

Navier–Stokes equations refer to time-dependent continuity second-order nonlinear partial differential equations that describe the motion of incompressible fluids and are the fundamental equations of fluid dynamics. The solutions may be used to model weather, water flow in a pipe, air flow around an airplane wing and many other things. Navier–Stokes equations can be written in $\mathbb{R}^n$ as follows

$$u_t + u \nabla u - \mu \Delta u + \nabla p = f$$
$$\nabla \cdot u = 0 ,$$

where $u : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the velocity of the fluid, $u_t$ the change of velocity with time, $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ the pressure, $\nabla p$ the pressure gradient ($\nabla$ is the Levi-Civita connection associated with the metric), $\Delta u$ the diffusion ($\Delta$ is laplacian); $f$ represents the external force that acts on the fluid (gravitational force or electromagnetic) and the convection acceleration term which is the nonlinear part is defined as

$$\text{div}(u \otimes u) = u \nabla u + \text{div}(u)u .$$

When the flow is slow, it is convenient to consider a linearized version of Navier–Stokes equations which are called Stokes equations. The study of the Stokes system is not only significant as such, but is also an essential starting point for the formation of nonlinear theory.
2.1.1 Navier–Stokes equations on Riemannian manifolds

Let \( M \) be a smooth orientable compact Riemannian manifold with or without a boundary of dimension \( n \), and let \( g \) be its Riemannian metric. If we consider the Laplacian as the \( \text{div grad} \) operator, then for any vector field \( u \), the \( \text{Bochner Laplacian} \) is defined by the formula

\[
(\Delta_B u)^k = \text{div}(g^{ij}u^k_j) = g^{ij}u^k_j
\]

and the \( \text{Hodge Laplacian} \) is given by the formula

\[
\Delta_H u = -\sharp(\delta d + d\delta)u.
\]

Let us further define \((2,0)\)-tensors

\[
(Su)^{kj} = g^{ki}u^j_i + g^{ij}u^k_j,
\]

\[
(Cu)^{kj} = (Su)^{kj} - \frac{2}{n} \text{div}(u)g^{kj},
\]

\[
(Au)^{kj} = g^{ki}u^j_i - g^{ij}u^k_j.
\] (2.1)

Above we view \( S, C \) and \( A \) as differential operators which operate on vector field \( u \).

Lemma 2.1.1

\[
Lu = \text{div}(Su) = \Delta_B u + \text{grad}(\text{div}(u)) + \text{Ri}(u).
\]

This lemma is recognized as one of the most significant results for writing the Navier–Stokes system on the Riemannian manifolds and plays a key role in this section.

Lemma 2.1.2

\[
L_C u = \text{div}(Cu) = \Delta_B u + (1 - \frac{2}{n})\text{grad}(\text{div}(u)) + \text{Ri}(u)
\]

\[
L_A u = \text{div}(Au) = \Delta_H u - \text{grad}(\text{div}(u)) = \Delta_B u - \text{grad}(\text{div}(u)) - \text{Ri}(u).
\]

Let us write \( S_u, C_u \) and \( A_u \) when we consider \( Su, Cu \) and \( Au \) as tensors of type \((1,1)\). Note that \( S_u = S_u^*, C_u = C_u^* \) and \( A_u = -A_u^* \).

Operator \( S \)

It will be useful to view \( S \) not only as a tensor but as a map, in two different ways:

(i) \( S \) can be considered as a differential operator which maps vector fields to tensor fields. In this case it is convenient to write \( Su \) as formula (2.1).

(ii) \( S \) induces a map \( S_u : T_p M \rightarrow T_p M \), given by the formula

\[
S_u v = (g^{ki}u^j_i + g^{ij}u^k_j)\delta^{\ell}_j v^{\ell} = (g^{ki}u^j_i g^{\ell}_j + u^k_j) v^{\ell}.
\] (2.2)
Note that $S_u$ is symmetric:
\[ g(S_u v, w) = g(v, S_u w) \]
for all vector fields $u$, $v$ and $w$.

Let us then introduce the corresponding bilinear map which is denoted by $b_S$:
\[ b_S(u, v) = (g^{ki} u^j g_{j\ell} + u^i) v_{\ell k}^j. \] (2.3)

The map $b_S$ is symmetric, i.e.
\[ b_S(u, v) = b_S(v, u) \]
and
\[ b_S(u, v) = g(\nabla u, \nabla v) + g((\nabla u)^T, \nabla v) \]

Remark 2.1.1 By definition, for any vector field $u$ and $v$ we obtain
\[ b_S(u, v) = \frac{1}{2} g(S_u, S_v) . \]

It is simple to check
\[ \text{div}(S_u v) = b_S(u, v) + g(L u, v) . \] (2.4)

Let us now take $L$ as our diffusion operator to formulate the homogeneous Navier–Stokes equations on Riemannian manifold as follows
\[
\begin{align*}
  u_t + \nabla_u u - \mu L u + \text{grad}(p) &= 0 \\ 
  \text{div}(u) &= 0 .
\end{align*}
\] (2.5)

In the above formulas, $\mu$ is the viscosity in the Navier–Stokes equations and indicates the amount of fluid resistance against the flow.


When an object moves in a flowing fluid, the fluid layers around it begin to resist the movement of the object so that the velocity of different layers vary. This fluid behavior depends on two factors: viscosity, the quantity that describes a fluid resistance to flow, and density. It is easy to prove that diffusion appears as a result of viscosity in Navier–Stokes equations.

**Operator $C$**

Operator $C$ can define a map $C_u : T_p M \to T_p M$, which is given by the formula
\[
C_u v = (g^{ki} u^j_k + g^{ki} u^j_k) g_{j\ell} v^{\ell} - \frac{2}{n} u^i_k g^{ji} g_{j\ell} v^{\ell} \\
= (g^{ki} u^j_k g_{j\ell} + u^i) v_{\ell k}^j . \] (2.6)

Let us then introduce the following bilinear map:
\[ b_C(u, v) = b_S(u, v) - \frac{2}{n} \text{div}(u) \text{div}(v) \\
= (g^{ki} u^j_k g_{j\ell} + u^i) v_{\ell k}^j - \frac{2}{n} u^i_k v_{\ell k}^j . \] (2.7)

We can confirm that
\[ \text{div}(C_u v) = \text{div}(S_u v) - \frac{2}{n} \text{div}(\text{div}(u)) v \\
= b_C(u, v) + g(L_C u, v) . \] (2.8)
Remark 2.1.2 By definition, for any vector field $u$ and $v$ we obtain
\[ b_C(u,v) = \frac{1}{2} g(C_u, C_v) . \]

Note that $b_C$ is symmetric, i.e.
\[ b_C(u,v) = b_C(v,u) . \]

Operator $A$

By considering operator $A$ in (2.5) on a Riemannian manifold, Hodge Laplacian could construct the system of Navier–Stokes as follows
\begin{align*}
    u_t + \nabla_u u - \mu \Delta_H u - 2\mu \text{Ri}(u) + \text{grad}(p) &= 0 \\
    \text{div}(u) &= 0 .
\end{align*}

(2.9)

2.2 DIFFERENT TYPES OF VECTOR FIELDS

This section deals with various types of vector fields on Riemannian manifolds, which can help to solve Navier–Stokes equations.

Killing vector field

The vector fields whose flows preserve the metric at any point have significant applications in mathematics and physics [13]. In honor of the German mathematician Wilhelm Killing, these vector fields are named Killing Fields. Killing vector fields (especially constant length) have been studied in many instances [19]. The geometry of the Riemannian manifolds that adopt the Killing vector fields has also been extensively studied. Given the non-trivial Killing vector fields on the Riemannian manifold, the Ricci curvature can’t be negative [13].

Definition 2.2.1 Vector field $u$ is Killing if
\[ S_u = 0 . \]

We may say in the same way that $u$ is Killing, if
\[ g(\nabla_v u, w) + g(\nabla_w u, v) = 0 , \]
for all vectors $v$ and $w$.

The following lemma is a practical lemma for solving Navier–Stokes equations.

Lemma 2.2.1 $b_S$ is positive semi-definite, i.e. $b_S(u,u) \geq 0$ for all $u$ and $b_S(u,u) = 0$ only if $u$ is a Killing vector field.

It should be noted that lemma 2.2.1 can be proven by considering the map $S_u$.

One can readily check that $b_S(u,u) = \frac{1}{2} \text{tr}(S_u^2)$. The map $S_u$ is symmetric, hence
\[ g(S_u^2 v, v) = g(S_u v, S_u v) \geq 0 , \]
and thus the eigenvalues of $S_u^2$ are non-negative. Since the trace is the sum of the eigenvalues this implies that $b_S$ is positive semi-definite. The equality is possible only if $S_u$ is the zero map, which occurs only if $u$ is a Killing vector field.
Lemma 2.2.2 For Killing vector field $u$ we obtain

$$\nabla_w(\nabla_v u) = -R(u, w)v$$

$$u^i_{;jk} = -u^i R^i_{;ikh}.$$  (2.11)

Proof. Suppose $u$ is a Killing vector field, then

$$u_{i;hk} + u_{h;jk} = 0.$$  

By applying the Ricci Identity and the above equation, we obtain

$$u_{i;hk} = -u_{h;ki} - u_{k;hi} = (u_{k;ih} + u_{i;R}^k_{hik}) - u_{i;R}^k_{ikh}$$

so that

$$2u_{i;hk} = u_{i;R}^k_{hik} - u_{i;R}^k_{ikh}$$

and, subsequently

$$u^i_{;hk} = -u^i R^i_{;ikh}.$$  

□

Conformal Killing vector field

A conformal Killing vector field is a specific Killing field generalization, but the Ricci tensor must not in this case be “so positive” [18]. If its flow consists of conformal transformations, the vector field is a conformal Killing field. We define a conformal Killing Vector $u$ as a vector field on a manifold so that its Lie derivative is proportional to itself if the metric is taken along the curves created by $u$:

$$\mathcal{L}_u g = \frac{2}{n} \text{div}(u) g.$$  

Definition 2.2.2 Vector field $u$ is conformal Killing if

$$Cu = 0.$$  

Lemma 2.2.3 $b_C$ is positive semidefinite, i.e. $b_C(u, u) \geq 0$ for all $u$ and $b_C(u, u) = 0$ only if $u$ is a conformal Killing vector field.

As in remark 2.1.2, one can easily check $b_C(u, u) = \frac{1}{2} \text{tr}(C^2 u)$. Therefore $b_C$ is positive semidefinite, the same as in the lemma 2.2.1 statement. When $u$ is a conformal Killing vector field $b_C(u, u) = 0$.

Harmonic vector field

Harmonic vector fields are another important group of vector fields that play an important role in this section. Harmonic vector fields on the Riemannian manifold $(M, g)$ were defined as the harmonic section of the Riemannian metric $g$ on $M$. Generally (but not always), harmonic vector fields are special in metrics. A vector field $u$ is said to be harmonic if it is in the kernel of the Laplace operator. i.e.

$$\frac{1}{2}(\delta d + d\delta)u = 0.$$  

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Definition 2.2.3  Vector field $u$ is harmonic if

$$Au = 0.$$ 

Then, $u$ is identified as a harmonic vector field if

$$g(\nabla u, w) - g(\nabla w, v) = 0$$

for all vectors $v$ and $w$.

It might be important to confirm that harmonic vector fields are solenoidal vector fields.

Theorem 2.2.4  [6] If $u$ is a Killing and $v$ a harmonic vector field, then $[u, v]$ is harmonic.

Parallel vector field

If some vector $u_p$ exists at $p \in M$ whose displacement vector at any point $q \in M$ does not depend on the particular smooth curve joining $p$ to $q$, the process yields a vector field on the manifold $M$ which we call a parallel vector field. The parallel fields on Euclidean space are all constant vector fields [11].

Definition 2.2.5  (Parallel Field, Petersen 1998, [13]) The vector field $u$ is parallel if and only if its covariant derivative in the direction of any vector field $v$ vanishes identically, $\nabla u = 0$. In general, let $\nabla$ be an affine connection on a manifold $M$, then a vector field $u$ is said to be parallel if $\nabla u = 0$.

2.3 SOME USEFUL FORMULAS

Here, we present many practical integrals and formulas which can permit us to go further in this section.

If $A$ is a $(2,0)$-tensor, then for any vector field $w$ we obtain

$$- \int_M g(\text{div} A, w) \omega = \int_M g(A, \nabla w) \omega - \int_{\partial M} g(A(\flat(w)), v) \omega_{\partial M},$$

where $\nu$ is defined as the outward unit normal vector field along $\partial M$. In particular, if $A = u \otimes v$, then

$$- \int_M g(\text{div}(u \otimes v), w) \omega = \int_M g(u \otimes v, \nabla w) \omega - \int_{\partial M} g(v, w) g(u, v) \omega_{\partial M}.$$ 

In particularly, the following lemma is functional:

Lemma 2.3.1  ([16], lemma 3.5) Let $u$ and $v$ be vector fields. Then

$$- \int_M g(\text{div} \nabla u, v) \omega = \int_M g(\nabla u, \nabla v) \omega - \int_{\partial M} g(\nabla v, v) \omega_{\partial M}$$

and, in general, we obtain

$$- \int_M g(\text{div} S u, v) \omega = \int_M b_S(u, v) \omega - \int_{\partial M} \left( g(\nabla v u, v) + g(\nabla v u, v) \right) \omega_{\partial M}.$$
The following inequality is useful in solving the Navier–Stokes equation and its applications are given in [16], page 5.

**Lemma 2.3.2** Let us consider a divergence free vector field $u$ on an unbounded Riemannian manifold $M$. We have

$$
\left| \int_M \text{Ri}(u,u) \omega_M \right| \leq \int_M g(\nabla u, \nabla u) \omega_M .
$$

Assume that $\text{div}(u) = 0$ on $M$. By applying lemma 2.1.1 and 2.3.1, one obtains

$$
\int_M g(\nabla u, \nabla u) \omega = \int_M b_S(u,u) \omega + \int_M \text{Ri}(u,u) \omega
$$

The first inequality is due to applying lemma 2.2.1 in the above equation, as follows

$$
\int_M \text{Ri}(u,u) \omega \leq \int_M g(\nabla u, \nabla u) \omega
$$

Moreover, by applying lemma 2.2.1 we have

$$
- \int_M g(\nabla u, \nabla u) \omega \leq \int_M \text{Ri}(u,u) \omega
$$

and the proof is completed.

The useful result of the above lemma is that

- $\int_M \text{Ri}(u,u) \omega_M = 0$ if $u$ is parallel,
- $\int_M \text{Ri}(u,u) \omega_M = \int_M g(\nabla u, \nabla u) \omega_M$ if $u$ is Killing,
- $\int_M \text{Ri}(u,u) \omega_M = - \int_M g(\nabla u, \nabla u) \omega_M$ if $u$ is harmonic.

**Lemma 2.3.3** The trilinear $T$ form is set to

$$
T(u,v,w) = \int_M g(\nabla u v, w) \omega ,
$$

where $M$ is an unbounded Riemannian manifold for any vector field $u$, $v$ and $w$. If we suppose $\text{div}(u)=0$, then

$$
T(u,v,v) = 0 .
$$

We have

$$
g(\nabla u v, v) = g_{hk} u^i v^j v^k = \frac{1}{2} u^i (g_{hk} v^j v^k) ; i = \frac{1}{2} g(u, \text{grad}(g(v,v))) ,
$$

so by definition

$$
T(u,v,v) = \int_M g(\nabla u v, v) \omega = \frac{1}{2} \int_M g(u, \text{grad}(g(v,v))) \omega
$$

$$
= \frac{1}{2} \int_M \text{div}(u g(v,v)) \omega - \frac{1}{2} \int_M \text{div}(u) g(v,v) \omega
$$

$$
= 0 .
$$

The following two lemmas will contribute very effectively to solving the Navier–Stokes problem.
Lemma 2.3.4 If \( w \) is Killing and \( \text{div}(u) = \text{div}(v) = 0 \), then
\[
\int_M (g(\nabla_v u, w) + g(\nabla_u v, w)) \omega_M = 0.
\]

Using lemma 2.3.3 and formula (2.10) we obtain
\[
\int_M (g(\nabla_v u, w) + g(\nabla_u v, w)) \omega_M = -\int_M (g(\nabla_v v, u) + g(\nabla_u u, v)) \omega_M = 0.
\]

Lemma 2.3.5 If \( w \) is harmonic and \( \text{div}(u) = \text{div}(v) = 0 \), then
\[
\int_M (g(\nabla_v u, w) - g(\nabla_u v, w)) \omega_M = 0.
\]

By applying definition 2.2.3, the proof of the above lemma can be obtained.

Next, we describe some more efficient formulas
\[
V_{\text{P}} = \{ u \in H^1(M) \mid \| u \| = 1, \langle u, v \rangle = 0 \text{ for all parallel } v \},
\]
\[
V_{\text{K}} = \{ u \in H^1(M) \mid \| u \| = 1, \langle u, v \rangle = 0 \text{ for all Killing } v \},
\]
\[
V_{\text{H}} = \{ u \in H^1(M) \mid \| u \| = 1, \langle u, v \rangle = 0 \text{ for all harmonic } v \}.
\]

We can then
\[
\alpha_{\text{P}} = \inf_{u \in V_{\text{P}}} \int_M g(\nabla u, \nabla u) \omega_M,
\]
\[
\alpha_{\text{K}} = \inf_{u \in V_{\text{K}}} \int_M g(S_u, S_u) \omega_M,
\]
\[
\alpha_{\text{H}} = \inf_{u \in V_{\text{H}}} \int_M \left( g(A_u, A_u) + \text{div}(u)^2 \right) \omega_M.
\]

Lemma 2.3.6 The numbers \( \alpha_{\text{P}}, \alpha_{\text{K}} \) and \( \alpha_{\text{H}} \) are strictly positive and have Poincaré type inequalities:
\[
\alpha_{\text{P}} \int_M g(u, u) \omega_M \leq \int_M g(\nabla u, \nabla u) \omega_M, \quad \forall u \in V_{\text{P}},
\]
\[
\alpha_{\text{K}} \int_M g(u, u) \omega_M \leq \int_M g(S_u, S_u) \omega_M, \quad \forall u \in V_{\text{K}},
\]
\[
\alpha_{\text{H}} \int_M g(u, u) \omega_M \leq \int_M \left( g(A_u, A_u) + \text{div}(u)^2 \right) \omega_M, \quad \forall u \in V_{\text{H}}.
\]

2.3.1 Operators rot, Rot and Curl

Let us now suppose that \( M \) is a two-dimensional Riemannian manifold. We can check \( \iota_w = * \circ b \), where \( b \) is the usual map \( T_p M \to T_p^* M \) defined by the Riemannian metric and * is the Hodge operator. The divergence operator is defined using the following diagram:
\[
\begin{align*}
\mathcal{X}(M) &\xrightarrow{\text{div}} V \xrightarrow{0} \\
\wedge^{n-1} M &\xrightarrow{\iota_w} \wedge^n M \xrightarrow{*} 0
\end{align*}
\]
so \( \text{div} = * \circ d \circ * \circ b \).

We define the operator \( \text{rot} \) by requiring that the following diagram commutes.

\[
\begin{array}{c}
0 \rightarrow V \xrightarrow{\text{grad}} \mathfrak{X}(M) \xrightarrow{\text{rot}} V \xrightarrow{} 0 \\
0 \rightarrow V \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{} 0 \\
\end{array}
\]

We then define the operator \( \text{Rot} \) by the following diagram.

\[
\begin{array}{c}
0 \rightarrow V \xrightarrow{\text{Rot}} \mathfrak{X}(M) \xrightarrow{\text{div}} V \xrightarrow{} 0 \\
0 \rightarrow V \xrightarrow{d} \Lambda^1 M \xrightarrow{d} \Lambda^2 M \xrightarrow{} 0 \\
\end{array}
\]

In the above diagrams we get

\[
\begin{align*}
\text{rot} \circ \text{grad} &= 0 \\
\text{div} \circ \text{Rot} &= 0 \\
\text{rot} \circ \text{Rot} &= -\text{div} \circ \text{grad} = -\Delta .
\end{align*}
\]

Let \( K = \iota_\omega^{-1} \circ b \) where the operator \( K \) rotates the vector field by 90 degrees. Hence, we can write

\[
\text{rot} = \text{div} \circ K \quad \text{and} \quad \text{Rot} = K \circ \text{grad} .
\]

Lemma 2.3.7

\[ K^2 = -\text{id} . \]

\textbf{Proof.} Since \( \iota_\omega = * \circ b \), we achieve \( \iota_\omega^{-1} = -* \circ * \) which, means that \( K = -* \circ * \circ b \). Then

\[
K^2 = * \circ * \circ b \circ * \circ * \circ b = * \circ * \circ b = -\text{id} .
\]

\[ \square \]

Lemma 2.3.8

\[
\begin{align*}
g(Ku, Kv) &= g(u, v) \\
g(Ku, v) &= -g(u, Kv) .
\end{align*}
\]

\textbf{Proof.} Let \( J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and let \( u \) be a vector field in some coordinates. We then have

\[
Ku = -\sqrt{\text{det}(g)} \ g^{-1} J u = \frac{-1}{\sqrt{\text{det}(g)}} J gu .
\]

Hence, for any vector field \( v \) we have

\[
\begin{align*}
g(Ku, v) &= -\sqrt{\text{det}(g)} \ g(g^{-1} Ju, g v) = \sqrt{\text{det}(g)} \ g(u, Jv) \\
&= \sqrt{\text{det}(g)} \ g(u, gg^{-1} Jv) = -g(u, Kv) .
\end{align*}
\]
Further,
\[ g(Ku, Kv) = -g(u, K^2v) = -g(u, -v) = g(u, v) \]

Thus we have the following formula:
\[
div(a Ku) = a \, div(Ku) + g(\text{grad}(a), Ku) \\
= a \, \text{rot}(u) - g(\text{Rot}(a), u) .
\]

In coordinates, we reformulate the above formulas as follows:
\[ \epsilon = \sqrt{\det(g)(dx_1 \otimes dx_2 - dx_2 \otimes dx_1)} . \]

Note that \( \nabla \epsilon = 0 \). Then the operator \( K \) can be defined as
\[ (Ku)^k = g^{ki} \epsilon_{ij} u^j ; \tag{2.12} \]

similarly, operator \( \text{rot} \) can be rewritten as
\[ \text{rot}(u) = div(Ku) = \epsilon^i_j u^j_k ; \tag{2.13} \]

For operator \( \text{Rot} \) also
\[ (\text{Rot}(u))^k = - (K \text{grad}(u))^k = - \epsilon_{ij}^k g^{ij} u^j = - \epsilon_{ji}^k u^j . \]

Let us suppose that \( M \) is a three-dimensional Riemannian manifold. Then we define the operator \( \text{curl} \) by requiring that the following diagram commutes.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & V & \xrightarrow{\text{grad}} & \mathcal{X}(M) & \xrightarrow{\text{curl}} & \mathcal{X}(M) & \xrightarrow{\text{div}} & V & \longrightarrow & 0 \\
0 & \longrightarrow & V & \xrightarrow{d} & \Lambda^1 M & \xrightarrow{d} & \Lambda^2 M & \xrightarrow{d} & \Lambda^3 M & \longrightarrow & 0 \\
\end{array}
\]

Let us now define
\[
\epsilon = \sqrt{\det(g)} \left( dx_1 \otimes dx_2 \otimes dx_3 - dx_2 \otimes dx_1 \otimes dx_3 - dx_3 \otimes dx_2 \otimes dx_1 \\
- dx_1 \otimes dx_3 \otimes dx_2 + dx_2 \otimes dx_3 \otimes dx_1 + dx_3 \otimes dx_1 \otimes dx_2 \right) .
\]

Again \( \nabla \epsilon = 0 \). We can then express \( \text{curl} \) in coordinates by the formula
\[ (\text{curl}(u))^k = \epsilon^i_j g^{ij} u^j_l . \]

We can also define the cross product of two vector fields by
\[ (u \times v)^{\ell} = (\sharp \ast (\eta u \wedge \eta v))^{\ell} = g^{\ell k} \epsilon_{ijk} u^i v^j . \]

### 2.4 NAVIER–STOKES SYSTEM ON TWO-DIMENSIONAL MANIFOLDS

The purpose of this section is to identify and mathematically justify the two-dimensional equations of a nonlinear Navier–Stokes system. The solution of our system concerns a case when \( R_i = \kappa g \), where \( \kappa \) is the Gaussian curvature. Hence, the system can be written as follows
\[
\begin{align*}
  u_t + \nabla_u u - \mu \Delta_B u - \mu \kappa u + \text{grad}(p) &= 0 \\
  - \Delta p + \text{tr}((\nabla u)^2) - \kappa g(u, u) + 2 \mu g(\text{grad}(\kappa), u) &= 0 \\
  \text{div}(u) &= 0 .
\end{align*}
\]
2.4.1 Navier–Stokes system with Coriolis effect

The Coriolis effect plays a significant role in the Navier–Stokes system. Let us consider the Coriolis effect in the unit sphere $S^2$ in $\mathbb{R}^3$. Let us choose the axis of rotation as the $x_3$ axis and let $\hat{\omega} = (0, 0, \omega)$ be the rotation vector which describes the rotation of the coordinate system. There are two new effects, the centrifugal and the Coriolis, which emerge from the rotation. The centrifugal effect at point $x \in \mathbb{R}^3$ is given by

$$ F_{\text{cen}} = \hat{\omega} \times (\hat{\omega} \times x) . $$

We choose to work with a modified pressure, allowing us to ignore the centrifugal effect.

Denoting the spherical coordinates by $(r, \theta, \phi)$ we have $x_3 = r \cos(\phi)$ and since $x$ and $u$ are orthogonal, the cross product is simply a rotation. By denoting $K(u)$ the vector field is obtained from $u$ by rotating $\frac{\pi}{2}$ radians in the tangent plane in a counterclockwise direction. We can write

$$ F_{\text{cor}} = a K(u) , $$

where $a = 2 \omega \cos(\phi)$. Note that the Coriolis effect is independent of the radius of the sphere. Then the Navier–Stokes system can be written as

$$ u_t + \nabla_u u - \mu Lu + a K u + \nabla p = 0 $$

$$ - \Delta p + tr((\nabla u)^2) - g(u, u) - \text{div}(a K u) = 0 \tag{2.14} $$

$$ \text{div}(u) = 0 . $$

Note that the second equation (known as the Poisson equation) for the pressure is obtained by taking the divergence of the first equation.

For instance, the variational formulation for the homogeneous Stokes system with a Coriolis effect is as follows:

$$ \mu \int_M b_S(u, v) + \int_M a g(Ku, v) + \int_M g(\nabla p, v) = 0 $$

$$ \int_M g(u, \nabla w) = 0 $$

for any vector field $v$ and $w$.

If $u$ is not a Killing vector field, we get the contradiction by choosing $v = u$, $w = p$ and $g(Ku, u) = 0$. Let us then suppose that $u$ is Killing, then

$$ a K(u) + \nabla p = 0 $$

and such $p$ exists if the compatibility condition

$$ \text{rot}(a K(u)) = 0 $$

holds. As in formula (2.13) we obtain

$$ \text{rot}(a K(u)) = \text{div}(a K^2(u)) = -\text{div}(a u) = -g(\text{grad}(a), u) . $$

As shown above $g(\text{grad}(a), u) = 0$. Thus the only Killing vector fields which satisfy the Stokes system with a Coriolis effect has the form $u = c \partial_\theta$, where $c$ is constant.
Remark 2.4.1 For any vector fields $u$ and $v$ on a two-dimensional manifold we have
\[
\nabla (Ku)v + \nabla_u (Kv) = \text{div}(v) Ku + \text{div}(Kv) u .
\]
In particular,
\[
\nabla_{Ku} Ku - \text{div}(Ku) Ku = \nabla_u u - \text{div}(u) u .
\]

Proof. First we have
\[
(Ku)^i = g^{ik} \epsilon_{kj} u^j = \epsilon_{12} (g^{k1} u^2 - g^{k2} u^1)
\]
\[
(\nabla_{Ku} v)^i = g^{ik} \epsilon_{kj} v^j_k = \epsilon_{12} (g^{k1} v^2_j - g^{k2} v^1_j)
\]
\[
(\nabla_u Kv)^i = g^{ik} \epsilon_{kj} v^j_k = \epsilon_{12} (g^{k1} v^2_k - g^{k2} v^1_k)
\]
\[
\text{div}(Kv) = g^{ik} \epsilon_{kj} v^j_k = \epsilon_{12} (g^{k1} v^2_k - g^{k2} v^1_k) .
\]
We must show that
\[
w^i = (g^{k1} u^2 - g^{k2} u^1) v^j_k + (g^{l1} v^2_l - g^{l2} v^1_l) u^k - (g^{k1} u^2 - g^{k2} u^1) v^j_k - (g^{k1} v^2_k - g^{k2} v^1_k) u^i = 0 .
\]
By simply expanding the components we can check that $w^1 = w^2 = 0 . \square$

In addition to fundamental interest, the subjects of vorticity and vortices are of general interest in mechanical engineering, chemical engineering as well as in powder technology. The studies of vortices also has a respected history. Leonardo da Vinci, for example, portrayed very interesting sketches of different sorts of vortices and eddy flows. The vorticity of the Navier–Stokes flow is scalar in the two-dimensional case, defined by
\[
\zeta = \text{rot}(u) = \text{div}(Ku) = \epsilon_{ij} u^j_i .
\]
This quantity corresponds to the rotation of the fluid. Flow without vorticity is called irrotational flow. Indeed, when the external force is only gravitational, the fluid flow describes an irrotational motion.

For example, for the Stokes flow on the sphere $S^2$ we can write
\[
\begin{align*}
\dot{u}_t - \mu \text{L}u + a K(u) + \text{grad}(p) = f \\
\text{div}(u) = 0 .
\end{align*}
\]
By using the above formula with rot we obtain:
\[
\zeta_t - \mu \text{rot}(L u) + \text{rot}(a K(u)) = \text{rot}(f) ,
\]
where as in lemma 2.1.1
\[
\text{rot}(L u) = \text{rot}(\Delta B u) + \text{rot}(\text{grad}(\text{div}(u))) + \text{rot}(\text{Ri}(u))
\]
when
\[
\text{rot}(\Delta B u) = \Delta \zeta + \kappa \zeta + g(Ku, \text{grad}(\kappa))
\]
\[
\text{rot}(\text{Ri}(u)) = \kappa \zeta + g(Ku, \text{grad}(\kappa)) .
\]
Eventually, in the two-dimensional case we obtain
\[
\text{rot}(L u) = \Delta \zeta + 2 g(Ku, \text{grad}(\kappa)) + 2 \kappa \zeta .
\]
In the three-dimensional manifold we have:
(i)  
\[
\text{curl}(\nabla u) = \nabla u\zeta - \nabla\zeta u + \zeta \text{div}(u). 
\]

Using the Ricci identity \((A.1)\) we get
\[
\text{curl}(\nabla u) = e^{ijk} g_{ji} u_{k;i} + e^{ijk} g_{ji} u_{k;j} u_{;i} \\
= e^{ijk} g_{ji} u_{k;i} + e^{ijk} g_{ji} u_{k;j} u_{;i} - e^{ijk} g_{ji} u_{k;h} R_{;hi} \\
= e^{ijk} g_{ji} u_{k;i} + \nabla_u \zeta - e^{ijk} u_{;h} R_{;hi}. 
\]

The direct computation indicates that
\[
e^{ijk} g_{ji} u_{k;i} = -\nabla_u \zeta + \zeta \text{div}(u),
\]
and \(e^{ijk} u_{;h} R_{;hi} = 0\) by formula \((A.7)\).

(ii)  
\[
\text{curl}(Lu) = L \zeta + 2 \text{curl}(R_i(u)) - 2 R_i(\zeta) .
\]

Recall that
\[
Lu = \Delta B u + \text{grad}(\text{div}(u)) + R_i(u)
\]
and according to the definition of curl, it follows that \(\text{curl} \circ \text{grad} = 0\). Then we compute
\[
\text{curl}(\Delta B u) = e^{\text{sm}} g_{sk} g^{ij} u_{;jk} \\
\Delta B \zeta = e^{\text{sm}} g_{sk} g^{ij} u_{;ji}.
\]

Let \(T = \Delta B \zeta - \text{curl}(\Delta B u)\), then the formula \((A.2)\) gives
\[
T = e^{\text{sm}} g_{sk} g^{ij} \left( u_{;jk} - u_{;jk} + u_{;ij} + u_{;ij} - u_{;h} R_{;hi} \right).
\]

The formulas \((A.7)\) and \((A.8)\) yield
\[
e^{\text{sm}} g_{sk} g^{ij} u_{;jk} R_{;hi} = -2 e^{\text{sm}} (R_{;k} g_{sk} u_{;ji} + R_{;k} u_{;ji} - \frac{R_{;sk}}{2} g_{sk} u_{;hi}) \\
= -2 e^{\text{sm}} (R_{;k} g_{sk} u_{;ji} + R_{;k} u_{;ji}) - R_{;sk}. 
\]

Accordingly,
\[
T = e^{\text{sm}} g_{sk} g^{ij} u_{;jk} R_{;hi} = -e^{\text{sm}} u_{;h} R_{;hi}.
\]

As a definition, however, we have
\[
\text{curl}(R_i(u)) = e^{ijk} g_{ji} (R_{;hj} u_{;i} + R_{;hi} u_{;j}) = e^{ijk} (R_{;hj} u_{;i} + R_{;hi} u_{;j}),
\]
which yields
\[
T + \text{curl}(R_i(u)) = e^{\text{sm}} g_{sk} g^{ij} u_{;jk} R_{;hi} + e^{\text{sm}} u_{;h} R_{;hi} + R_{;sk}. 
\]

Then the direct computation shows that
\[
R_i(\text{curl}(u)) = e^{\text{sm}} g_{sk} g^{ij} u_{;jk} R_{;hi} + e^{\text{sm}} u_{;h} R_{;hi} + R_{;sk}. 
\]
3 Partial differential equations (PDEs) on Riemannian manifolds

We live in a world of constantly changing phenomena, and most of these changes can be described using differential equations. For example, to explain gravitational force, Albert Einstein used differential equations. In these equations, he explained this force and proved it was feasible to go into the future. Hence, differential equations are equations describing the physical, chemical and thermal behavior of a system under initial conditions called boundary conditions. In general, differential equations are divided into two types:

- Ordinary Differential Equations, known as ODE.
- Partial Differential Equations, known as PDE.

In summary, the ODE system is considered to be an equation having a single independent variable function with its derivatives. However, the PDE system is applied to a set of differential equations involving the unknown functions of several independent variables and the derivative of functions in relation to them.

In this chapter, we work with Killing and conformal Killing vector fields. We do not really know any general numerical programs in the fields for computing these vector fields, so we propose a method to compute these vector fields by reducing the problems to a symmetric eigenvalue problem. The Killing and conformal Killing vectors then appear as an elliptical operator's eigenspace that corresponds to the zero eigenvalues. We have used the program FREEFEM++ in the numerical solution of our eigenvalue problem.

3.1 PDE ON RIEMANNIAN MANIFOLDS

Most of the differential geometry problems on Riemannian manifolds can be reduced to differential equation problems. Our main purpose here is to analyze these equations on specific Riemannian manifolds such as Enneper’s surface, torus and the Klein bottle.

We have a manifold with a boundary and a single coordinate chart on the Enneper’s surface, so that the problem can be formulated in FREEFEM++ in the standard way. The torus is a nontrivial manifold formed by the revolution of a circle around a line of its plane; thus, it is a tube with a constant diameter and a circular bore. Solving problems analytically on the torus means looking for periodical solutions. Numerically, so-called periodic boundary conditions should take this into account, and they are also implemented in FREEFEM++.

The Klein bottle needs four-dimensions because lacking a hole, the surface will move through itself. In this sense, a Klein bottle is non-orientable two-dimensional manifold that can only exist through four dimensions. One can use a single coordinate domain, but here the domain boundary identifications are non-standard and
FREEFEM++ can’t be used. In this case, we apply the necessary identifications directly with C++ and compile the corresponding matrices.

3.1.1 A surface of revolution

Minimal surfaces (a surface that minimizes its own area locally) play an important role in math, physics, biology, design, and so on. Mathematicians and geometers have studied such surfaces over the years [10] and [14]. A minimal surface in \( \mathbb{R}^3 \) is a regular surface where the mean curvature vanishes identically. One of the easiest and most interesting types of minimal surfaces is that which refers to surfaces of revolution. A surface of revolution is a surface in Euclidean space \( \mathbb{R}^3 \) created by rotating a curve around an axis of rotation. Let the \( z \)-axis be the axis of revolution for the construction of this surface and use a parametric equation to build a revolutionary surface. It is important to understand first how a circle is formed in the plane because the surface of revolution consists of a set of circles at different heights. That is, if a plane parallel to the \( xy \)-plane cuts the surface, the intersection is a circle. For a circle of radius \( r \) in the \( xy \)-plane, the parametric equation is \( (x(x_2), y(x_2)) = (r \cos(x_2), r \sin(x_2)) \), where \( 0 \leq x_2 \leq 2\pi \). In a revolutionary surface, the radius may change at each height. If the height \( x_1 \) radius is \( c_1(x_1) \), then the surface equation is

\[
\phi(x_1, x_2) = (c_1(x_1)\cos(x_2), c_1(x_1)\sin(x_2), x_1),
\]

The standard parameterization of a surface of revolution is determined by

\[
\phi(x) = \begin{pmatrix} c_1(x_1)\cos(x_2) \\ c_1(x_1)\sin(x_2) \\ c_2(x_1) \end{pmatrix},
\]

where \( c(x_1) = (c_1(x_1), c_2(x_1)) \) is known as the profile curve.

For instance, a line revolution surface with an orthogonal line as a revolutionary axis is a plane, a surface of revolution of a circle with a circle diameter as a revolutionary axis is a sphere and the torus is the surface created by a circle revolution around a line of its axis; so it is thus a tube with a constant diameter and a circular bore.

**Enneper surface**

The literature includes many important classical works on minimal surfaces. One is the Enneper surface, stated by Alfred Enneper in 1864, which is an example of a negative curvature surface model. All in all, an Enneper surface is a simple connected minimal surface in \( \mathbb{R}^3 \) and supported by the map \( \phi : \mathbb{R}^2 \to \mathbb{R}^3 \) as follows:

\[
\phi(x) = \begin{pmatrix} x_1 - \frac{1}{3}x_1^3 + x_1x_2^2 \\ -x_2 + \frac{1}{3}x_2^3 - x_1^2x_2 \\ x_1^2 - x_2^2 \end{pmatrix}
\]

**Torus**

Topologically, a torus is a closed surface, two-dimensional compact manifold, defined as the Cartesian product of two circles in a specific \( S^1 \times S^1 \), and is a subset of
the 3-sphere $S^3$ of radius $\sqrt{2}$.

An interesting feature of a torus is that it is surface "connected". A torus is a revolutionary surface which is a three-dimensional geometric form that is formed by the rotation of a circle around an axis but non-tangent to that circle. The generator is called the circle and has the radius $r$. The distance between the generator’s axis and center is $R$.

Parametrically, a torus can be defined by the map $\varphi : U \subseteq \mathbb{R}^2 \to \varphi(U) \subseteq \mathbb{R}^3$ as below:

$$
\varphi(x) = \begin{pmatrix}
(R + r\cos(x_1))\cos(x_2) \\
(R + r\cos(x_1))\sin(x_2) \\
rsin(x_1)
\end{pmatrix},
$$

where $U = [0, 2\pi] \times [0, 2\pi]$. The metric is given by

$$
g = (d\varphi)^T (d\varphi) = \begin{pmatrix}
1 & 0 \\
0 & (R + \cos(x_1))^2
\end{pmatrix}.
$$

Klein bottle

In 1882, the German mathematician Felix Klein, a member of the Berlin Academy of Sciences, designed an interesting example of the compact non-orientable unbounded two-dimensional manifold known as the Klein bottle. The surface of this bottle is completely closed. However, we cannot know its internal or external surface; in other words, its volume is zero. Note that the Klein bottle does not intersect itself. This bottle can be rotated to any side without causing any liquid to enter it. Actually, the Klein bottle is a mathematical model, not a physical object. A Klein bottle can be embedded in $\mathbb{R}^4$, where it can be seen in three-dimensional space due to the restriction of disruption.

Common Klein bottle parameterization is defined by the map $\varphi : U \subseteq \mathbb{R}^2 \to \varphi(U) \subseteq \mathbb{R}^3$ below:

$$
\varphi(x) = \begin{pmatrix}
(2 + \cos(x_1))\cos(x_2) \\
(2 + \cos(x_1))\sin(x_2) \\
\sin(x_1)\cos(\frac{x_2}{2}) \\
\sin(x_1)\sin(\frac{x_2}{2})
\end{pmatrix},
$$

where $U = [0, 2\pi] \times [0, 2\pi]$. The metric is granted by

$$
g = (d\varphi)^T (d\varphi) = \begin{pmatrix}
1 & 0 \\
0 & \frac{1}{4}(3\cos^2(x_1) + 16\cos(x_1) + 17)
\end{pmatrix}.
$$

3.2 EIGENVALUE PROBLEM

Eigenvalues and eigenvectors are important and practical topics in linear algebra that have various applications in different fields such as computational fluid dynamics, elasticity theory, data science, and machine learning and control systems. They allow us to "minimize" a linear operation to separate and simplify the problem.

Definition 3.2.1 Given $n \times n$ matrix $A$ to find scalar $\lambda$ and nonzero vector $x$. The standard eigenvalue problem is

$$
Ax = \lambda x,
$$

where $\lambda$ is the eigenvalue, and $x$ is the corresponding eigenvector.

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Equation $Ax = \lambda x$ is equivalent to

$$(A - \lambda I)x = 0,$$

which has a non-zero $x$ solution if and only if its matrix is singular. Eigenvalues of $A$ are roots $\lambda_i$ of the characteristic polynomial

$$\det(A - \lambda I) = 0$$

in $\lambda$ of degree $n$.

### 3.2.1 Eigenvalue problem on Riemannian manifold

Let $(M, g)$ be a complete unbounded $n$-dimensional Riemannian manifold. We are interested in the eigenvalue problem for the Laplacian PDE on $M$, i.e. finding the $(\lambda, u)$ pairs, where $\lambda$ is a (real) number and $u$ a non-zero function, such that

$$-\Delta u = \lambda u .$$

(3.2)

**Remark 3.2.1** We define the inner product of two vector fields $u$ and $v$ on $M$ as follows

$$\langle u, v \rangle = \int_M g(u, v) \omega_M ,$$

where $\omega_M$ is the volume form; then the familiar Sobolev inner product can be defined as

$$\langle u, v \rangle_{H^1} = \int_M \left( g(u, v) + g(\nabla u, \nabla v) \right) \omega_M .$$

Using this notations, the eigenvalue problem (3.2) can be stated as follows. Find vector field $u$ and real number $\lambda$ so that

$$a(u, v) = \lambda \langle u, v \rangle \quad \forall v \in \tau(M) ,
$$

where

$$a(u, v) = \int_M g(\nabla u, \nabla v) \omega_M .$$
4 Special classes of Riemannian manifolds

In recent years, major steps have been made in understanding the relations between the local and global Riemannian manifold structures. There have been numerous categorized outcomes for different Riemannian manifold classes: for example manifolds with more geometrical structures, manifolds with curvature criteria, symmetric Riemannian spaces.

There are many interesting results on Riemannian manifolds which are mainly concerned those that satisfy the curvature conditions. For instance, manifold with constant sectional curvature and Einstein manifold have a prominent role to play in studying Riemannian geometry [1] and [5]. At the end of the 19th-century mathematicians wanted to classify all the local symmetric Riemannian manifolds after studying constant curvature spaces. A symmetric space is a Riemannian manifold $(M, g)$ that for every point $p \in M$ there exists an isometry $\sigma_p$ of the same $(M, g)$, called an involution, such that:

$$\sigma_p(p) = p$$
$$d\sigma_p = -id_{T_pM} .$$

Cartan began to study the symmetry of a manifold [2], but his idea has been challenged by various authors who have been given some curvature restrictions in different directions with several defining conditions. Cartan initially classified only connections of locally symmetrical spaces to the Riemannian space. Consequently, different weaker symmetries have been studied, such as recurrent manifolds by Yano [8], Ricci recurrent manifolds [12] and pseudo Ricci symmetric manifolds [9].

A (pseudo)-Riemannian manifold is $k$-symmetric if the following conditions are stated:

$$\nabla^k R = 0 , \quad \nabla^{k-1} R \neq 0 ,$$

where $k \geq 1$. Thus, a Riemannian manifold is symmetric when its curvature tensor satisfies $\nabla R = 0$.

This chapter proposes a unified approach to several classes of Riemannian manifolds, Ricci recurrent, pseudo Ricci symmetric, Cotton and quasi Einstein manifolds. The examples of these classes are defined by requiring the appropriate Ricci tensor to satisfy certain conditions $P$. Consequently, we will solve the systems of over-determined PDE.

4.1 INTERACTIONS BETWEEN VARIOUS CLASSES OF RICCI CURVATURE

In this section, we will consider several classes of Riemannian manifolds whose Ricci curvature is not zero. Such classes are defined by the fact that the corresponding Ricci tensor is satisfied by some condition $P$, which is a partial differential equation and helps to evaluate the metric on the manifold. We should, therefore, specify the class of $P$ for this purpose. In this case, the Ricci curvature is nicely represented below.
4.1.1 Ricci recurrent manifold

A manifold \((M, g)\) is named Ricci recurrent, RR, if

\[ R_{ij\ell} = \beta_{\ell} R_{ij} , \]  

(4.1)

where \(\beta\) is one form defining on the manifold.

Remark 4.1.1 A \((0, k)\)-tensor field \(T\) is said to be recurrent [15] if there exists a unique one-form \(\beta\) satisfying

\[ T_{i_1i_2 \cdots i_k\ell} = \beta_{\ell} T_{i_1i_2 \cdots i_k} . \]

We now present two ways to define one-form \(\beta\).

- Using trace (4.1) one-form \(\beta\) is set as

  \[ \beta_{\ell} = \frac{sc_{\ell}}{sc} = (\nabla \ln(sc))_{\ell} , \]  

  (4.2)

  if the scalar curvature is not zero.

- Multiplying (4.1) by \(R_{ij}^{\ell}\) we obtain

  \[ \beta_{\ell} = \frac{R_{ij}^{\ell} R_{ij\ell}}{|R|^2} = \frac{1}{2} (\nabla \ln(|R|^2))_{\ell} . \]  

  (4.3)

Example

Here we have access to the appropriate example, given by the proper metric that has been characterized on the manifold. The question is how to characterize this metric on a manifold?

By applying (4.2) on formula (4.1), we find a metric \(g\) such that \(P_{ijk} = 0\) where

\[ P_{ij\ell} = sc R_{ij\ell} - sc_{\ell} R_{ij} . \]  

(4.4)

The above system is a third-order quasi-linear of \(\frac{1}{2} n^2 (n + 1)\) PDE. Remember that this number of equations is the maximum number of PDE in the original system which are algebraically independent.

Remark 4.1.2 First-order PDEs are typically classified as linear, quasi-linear, or nonlinear.

- A first-order PDE is said to be linear if an unknown function \(u(x, y)\) is represented as the form

  \[ a(x, y) \frac{\partial u(x, y)}{\partial x} + b(x, y) \frac{\partial u(x, y)}{\partial y} + c(x, y) u(x, y) = d(x, y) . \]

- A first-order PDE is said to be quasi-linear if defined as follows:

  \[ a(x, y, u(x, y)) \frac{\partial u(x, y)}{\partial x} + b(x, y, u(x, y)) \frac{\partial u(x, y)}{\partial y} = c(x, y, u(x, y)) . \]

- A first-order nonlinear PDE is said to be neither linear nor quasi-linear.
Example 4.1.1 Consider the following metric in three-dimensional space:

\[ g = f_1(x^1)(dx^1)^2 + f_2(x^1)h_2(x^2)(dx^2)^2 + f_3(x^1)h_3(x^3)(dx^3)^2, \]  

(4.5)

where \( f_1(x^1), f_2(x^1), h_2(x^2), f_3(x^1), h_3(x^3), q(x^3) \neq 0 \). There are \( 18 = \frac{1}{2}(3)(4) \) independent equations in our PDE system (4.4), but by citing Maple there are probably just 14 non-zero equations. We also suppose \( sc_{hf} \neq 0 \). Therefore, the system is split into seven subsystems. Nonetheless, six systems either allow such unknown functions to be constant or the solutions give only a trivial solution in which \( \beta \) is reduced to zero. By computing with \( \text{rifsimp} \) there are three differential equations in the remainder of the system:

\[
\begin{align*}
  f''_2 &= \frac{f'_2(f'_1f_2 + f'_2f_1)}{2f_1f_2} \\
  f''_3 &= \frac{f_2f_3f'_1f'_3 + 2f_1f_2(f'_3)^2 - f_1f_3f'_2f'_3}{2f_1f_2f_3} \\
  h''_3 &= \frac{2f_1f_2f_3f'_3h_2(h'_3)^2 + f_1f_2f_3f'_3h_3h'_3}{2f_1f_2f_3f'_3h_2h_3} - \frac{f_1f_2^2(f'_3)^2h_2(h'_3)^2}{2f_1f_2f_3f'_3h_2h_3}.
\end{align*}
\]

(4.6)

Of course, if we consider \( f_1 \) as an arbitrary function in the above system, we have the first two equations:

\[
\begin{align*}
  f_1 &= f_1 \\
  f_2 &= c_3 \\
  f_3 &= c_2 e^{c_1(f \sqrt{f_1} dx^1)}.
\end{align*}
\]

Nonetheless, by resolving \( f_1 \) and \( f_3 \) with respect to \( f_2 \), one can actually provide the following family of solutions:

\[
\begin{align*}
  f_1 &= \frac{c_2(f'_3)^2}{f_2} \quad \text{and} \quad f_3 = c_1 f_2^m.
\end{align*}
\]

where \( m \) is an integer. We have the third differential equation in the system (4.6) which contains \( h_i \) and \( f_i \). If we replace the above formulas there, the \( f_i \) functions disappear and the remainder is

\[
\begin{align*}
  h''_3 &= \frac{(2m - 1)c_2h_2(h'_3)^2 + mc_2h_3h'_2h'_3 - m^2h_2^2h_3^2}{2mc_2h_2h_3}.
\end{align*}
\]

A new function \( h_3 = h^m \) is then introduced. For \( h_2 \) the result for the above equation is

\[
\begin{align*}
  h_2 &= \frac{c_2h_1^{1/m}(h'_3)^2}{h_3^2(c_2c_3 - m^2h_3^{1/m})} = \frac{c_2m^2(h'_3)^2}{h(c_2c_3 - m^2h)}.
\end{align*}
\]

We can then write our final metric by considering \( f \) instead of \( f_2 \)

\[
\begin{align*}
  g &= \frac{c_2(f'_3)^2}{f} (dx^1)^2 + \frac{m^2c_2f(h'_3)^2}{h(c_2c_3 - m^2h)} (dx^2)^2 + c_1 f^m h^m q(dx^3)^2.
\end{align*}
\]

One can obviously choose constants and functions to insure that \( g \) is positive definite and

\[
\begin{align*}
  sc = (1 - m)c_3/(2mfh) \quad \text{and} \quad \beta = -\nabla \ln(fh).
\end{align*}
\]

Note that if \( m = 0 \) then \( Ri = 0 \).
4.1.2 Pseudo Ricci symmetric

A pseudo Riemannian manifold is a differentiable manifold with a non-degenerate, symmetric and smooth metric tensor. Manifold \((M^n, g)\) \((n > 2)\) is called a pseudo Ricci symmetric, PRS, if there is a non-zero one-form \(\alpha\) such that

\[
R_{ij\ell} = 2\alpha_j R_{ij} + \alpha_i R_{ij} + \alpha_j R_{i\ell} .
\]

(4.7)

Remark 4.1.3 Multiplying (4.7) first in \(g^{ij}\) and second in \(g^{i\ell}\), we get

\[
sc_{\ell\ell} = 2\alpha_\ell sc + 2\alpha^i R_{i\ell}
\]

\[
sc_{ij} = 2\alpha_j sc + 6\alpha^i R_{ij}.
\]

(4.8)

Using the above equations we can obtain the most significant property in a PRS manifold such that \(\alpha^h R_{hi} = 0\).

We now have two ways to define one-form \(\alpha\).

- By applying remark (4.1.3) in (4.8), we get

\[
\alpha_\ell = \frac{1}{2} \frac{sc_{\ell}}{sc} = \frac{1}{2} \left(\nabla \ln(sc)\right)_\ell ,
\]

if the scalar curvature is not zero.

- Multiplying (4.7) in \(R^{ij}\) while using remark (4.1.3), yields

\[
\alpha_\ell = \frac{1}{2} \frac{R^{ij} R_{ij\ell}}{|R|^2} = \frac{1}{4} \left(\nabla \ln(|R|^2)\right)_\ell .
\]

(4.10)

Example

By applying formula (4.9) on (4.7) we can find a metric \(g\), that makes \(Q_{ijk} = 0\).

\[
Q_{ij\ell} = 2sc R_{ij\ell} - 2sc_j R_{ij} - sc_\ell R_{ij} - sc_j R_{i\ell} .
\]

(4.11)

This is a third-order quasi-linear system of \(\frac{1}{2} n^2(n + 1)\) PDE.

Example 4.1.2 In PDE system (4.11) and metric (4.5), we have only 14 non-zero equations and by citing Maple and computing with \texttt{risimp} we obtain three cases in which \(\alpha \neq 0\). The following is one case:

\[
f''_2 = \frac{f_2 f'_3}{f_3}
\]

\[
f''_3 = \frac{f_1 (f'_2)^2 + f_3 f'_1 f'_3}{2 f_1 f_3}
\]

\[
h''_3 = \frac{2 h_2 h_3 (h''_2 h'_3 + 3 h'_2 h''_3) + 4 h_2^2 h_3 h'_3 h''_3 - h_2^2 (h'_3)^2 - 2 h_2 h_3 h'_3 (h'_3)^2 - 2 h_2^2 (h'_3)^2 h''_3}{2 h_2^2 h_3^2}.
\]
We can solve \( f_1 \) and \( f_2 \) as regards \( f_3 \) in the first two equations, and \( h_2 \) in the last equation with respect to \( h_3 \):

\[
\begin{align*}
  f_1 &= \frac{c_1(f'_3)^2}{f_3} \\
  f_2 &= c_2 f_3 \\
  h_2 &= \frac{(h'_3)^2}{h_3(c_3 h_3 + c_4)}.
\end{align*}
\]

We presume \( f_3 = f \) and \( h_3 = h \), yielding the metric

\[
g = \frac{c_1(f')^2}{f} (dx^1)^2 + \frac{c_2 f(h')^2}{h(c_3 h + c_4)} (dx^2)^2 + fhq(dx^3)^2
\]

and the one-form \( \alpha \) and scalar curvature as

\[
\alpha = -\frac{f'}{2f} dx^1, \quad Ri = -\frac{c_1 c_3 + c_2}{4c_1 c_2 h} \left( \frac{c_2(h')^2}{c_3 h + c_4} (dx^2)^2 + qh^2(dx^3)^2 \right).
\]

### 4.1.3 Cotton

Throughout modern mathematics and theoretical physics, the Weyl tensor and the Cotton tensor have played a major role. The Weyl tensor emerges as an invariant of conformal metric tensor transformations. The vanishing of the Weyl tensor is an important result of Riemannian geometry as it is equivalent to the local conformal flatness of the manifold. However, this does not hold in three dimensions because the Weyl tensor has, in fact exactly vanished. There is another known tensor, the Cotton tensor, which obstructs the conformal local flatness of a pseudo-Riemannian (three-dimensional) manifold. In particular, it has conformal invariances in three dimensions, which are very close to those of the Weyl tensor.

**Definition 4.1.3** Let \( M \) be a \( n \)-dimensional Riemannian manifold with metric \( g \). Then

(i) The Schouten tensor is

\[
S_{ij} = Ri_{ij} - \frac{sc}{2(n-1)} g_{ij}
\]

(ii) The Cotton tensor is

\[
C_{ijk} = S_{ij;k} - S_{ik;j}
\]

(iii) The Weyl tensor is

\[
W_{hijk} = R_{hijk} + \frac{sc}{(n-1)(n-2)} \left( g_{hk} g_{ij} - g_{hj} g_{ik} \right)
- \frac{1}{n-2} \left( R_{ijk} g_{hi} - R_{ih} g_{ij} + R_{ij} g_{hk} - R_{ik} g_{hi} \right).
\]

A significant approach behind these definitions is that the trace-less part of curvature \( R_{hijk} \) is the Weyl tensor, \( W_{hijk} \), such that \( W_{kij}^E = 0 \).
Example

The Ricci curvature is Cotton, CO, if the Cotton tensor \( C = 0 \). This is the third-order quasi-linear system of \( n^2(n-1) \) PDE.

Example 4.1.4 Let us take PDE system \( C_{ijk} = S_{ijk} - S_{ikj} \) and metric (4.5). Then by citing Maple our system has 8 equations. In the most common case, we get three cases when computing with \( \text{rifsimp} \)

\[
\begin{align*}
  f''_2 &= F(f_1,f_2,f_3) \\
  h''_3 &= H(f_1,f_2,f_3,h_2,h_3),
\end{align*}
\]

where \( F \) and \( H \) are very complicated, including the derivatives of their cases so that they are not explicitly written. The first equation can actually be solved by \( f_3 \) using quadratures yielding

\[
f_3(x^1) = \exp\left(\hat{F}(f_1,f_2,f'_1,f'_2,f''_2)\right),
\]

where \( \hat{F} \) also includes those integrals which are integral according to the specified variables. Now replace the second equation with the expression

\[
2h_2h_3h'_3 - h_3h'_2h'_3 - 2h_2(h'_3)^2 + c_0h_2^2h_3^2 = 0,
\]

where \( c_0 \) is constant. This can be overcome by

\[
h_2 = \frac{(h'_3)^2}{(c_1 - c_0 \ln(h_3))h_3^2}.
\]

Thus \( f_1, f_2, h_3 \) and \( q \) are available as a free functions in such a way that the metric is positive-definite.

4.1.4 Quasi Einstein

Definition 4.1.5 The Einstein manifold is a (pseudo-)Riemannian manifold, whose Ricci tensor is proportional to its metric tensor [1], i.e.,

\[
R_{ij} = a g_{ij}.
\]

From the analytical perspective, the Einstein equation is nonlinear of \( \frac{1}{2}n(n+1) \) second-order partial differential equations.

In 1977 [17], the concept of a quasi Einstein manifold, QE, was introduced. A non-flat Riemannian manifold \( (M^n,g) \) \( (n > 2) \) is known as a quasi Einstein manifold when its Ricci tensor is not identical to zero and follows the condition

\[
R_i = a g + b \omega \otimes \omega, \quad \text{ (4.12)}
\]

where \( a, b \) are scalars where \( b \neq 0 \) and \( \omega \) is a non-zero 1-form. If \( b = 0 \), the manifold is Einstein.

Remark 4.1.4 A pseudo Ricci symmetric manifold with the \( \text{div}(R) = 0 \) (resp. \( C = 0 \)) property is an Einstein manifold [7].
**Some Facts From Linear Algebra**

Let the tensor $T$ be introduced as $T = R_{ij} - a g_{ij}$. Suppose $W$ to be the vector space of real symmetric $n \times n$ matrices and let $V_r$ be the subvariety of symmetric matrices of a rank at most $r$. This is a determinantal variety, defined by setting to zero all minors of the size $(r+1) \times (r+1)$. There are thus $\binom{n}{r+1}$ polynomials which define $V_r$. However, $\text{codim}(V_r) = \binom{n-r+1}{r}$ gives the number of independent equations.

If the symmetric tensor $T$ is of matrix rank one, then all $2 \times 2$ minors are zero. Since there are $\binom{n+1}{2}$ independent components in $g$ and we get only $\binom{n}{2}$ independent differential equations, then $\binom{n+1}{2} - \binom{n}{2} = n$ functions remain, which can be chosen arbitrarily. In addition, there is the free function $a$. Consequently, it should be straightforward to construct examples.

Note, however, that the QE system is essentially different from the RR and PRS systems as a PDE system. QE is a fully nonlinear system, i.e. nonlinear in the highest derivatives while RR and PRS systems are quasi-linear.

Once the appropriate $T$ is found $b$ and $\omega$ can easily be computed. Multiplying the formula (4.12) by $g^{ij}$ we see that $sc = na + b$, which indicates that $b$ and $\omega$ can be solved from the linear system

$$T_{ij}\omega^j = (sc - na) \omega_i.$$ 

**Example**

We seek a metric $g$ and function $a$ such that the matrix rank of $T = R - g$ becomes one.

**Example 4.1.6** Let us take PDE system $T_{ij} = R_{ij} - a g_{ij}$ and metric (4.5). This yields only 5 non-zero equations. Remember that depending on the choice of $a$, we get different families of solutions, but just one set of solutions will be studied.

In this phase, we try to identify function $a$. To that end, we specify the minors of matrix $T$. The following results are then given by Maple:

$$a = A(f_1, f_2, f_3, h_2, h_3).$$

As $b = sc - 3a$, then

$$b = B(f_1, f_2, f_3, h_2, h_3).$$

After choosing $a$ and $b$ we are thus left with a single PDE; rifsimp then gives us the following system:

$$f_2''' = F_1(f_1, f_2, f_3)$$

$$f_3''' = F_2(f_1, f_2, f_3)$$

$$h_3''' = H(f_1, f_2, f_3, h_2, h_3).$$

The first two equations can be directly solved by denoting $f_2 = f$ as follows:

$$f_1 = \frac{c_1 c_3 (f')^2}{f_3 f}$$

$$f_3 = c_3 m^{-m} f_1^{-m} (c_2 f - 1)^m.$$
Replacing this with third equations
\[
h_3'' = \frac{(3m - 2)h_2h_3' + 2(m - 1)h_3h_2')h_3'}{4(m - 1)h_2h_3}.
\]
Denoting \( h_3 = h \) and solving the above equation for \( h_2 \) yields
\[
h_2 = c_4h^{(2 - 3m)/(2m - 2)}(h')^2.
\]
Subsequently, \( a, b \) and \( \omega \) are easily calculated.
\[
a = \frac{m}{8(1 - m)c_4f}h^{(2 - m)/(2m - 2)} - \frac{c_4^2 f^2 + (m - 2)c_2 f + (m - 1)^2}{2c_1m^m f^m} (c_2 f - 1)^{m - 2}
\]
\[
b = \frac{m}{8(m - 1)c_4f}h^{(2 - m)/(2m - 2)} + \frac{m(m - 1)}{2c_1m^m f^m} (c_2 f - 1)^{m - 2}
\]
\[
\omega = fh'(c_2 f - 1)\partial_{x_1} + (2 - 2m)hf'\partial_{x_2}.
\]
5 Summary of papers

The notation used in the original papers is modified to refer to the previous parts in the summaries below.

5.1 SUMMARY OF PAPER I

We demonstrate that the Killing vector fields play an important role in the study of the Navier–Stokes system. Several findings are also presented and a linear version of the Navier–Stokes system is evaluated which is of interest for numerical applications. We look at the case with two-dimensional properties and also consider the Coriolis effect that is important for the atmospheric flows.

5.1.1 Solutions to Navier–Stokes system and the Killing fields

Specific fluid properties can be calculated by using the Navier–Stokes equations. Since nonlinear terms are present in this system, numerical methods are typically used to solve them. Therefore, our system can be analytically solved only with some simplifications. In this section, we explore many of these facilitations. We first investigate different solution properties for the Navier–Stokes system. Then we continue by demonstrating that any solution can be broken down as $u = u^K + u^\perp$, where $u^K$ is Killing and $u^\perp$ is orthogonal to Killing fields. One can consider $u^K$ as a projection of the initial condition into the Killing fields space.

The Navier–Stokes equations have been defined on an arbitrary manifold $(M, g)$ as follows:

$$
\begin{align*}
\frac{du_t}{dt} + \nabla_u u - \mu Lu + \text{grad}(p) &= 0 \\
\text{div}(u) &= 0,
\end{align*}
$$

where $L$ is our diffusion operator and is defined in lemma 2.1.1.

By considering both formulas (5.1) and (2.9), it would be easy to check:

(i) if $u$ is parallel and $p$ is constant, then $(u, p)$ is a solution of the Navier–Stokes equations with Bochner Laplacian.

(ii) if $u$ is Killing and $p = \frac{1}{2} g(u, u)$, then $(u, p)$ is a solution of (5.1).

(iii) if $u$ is harmonic and $p = -\frac{1}{2} g(u, u)$ then $(u, p)$ is a solution of the Navier–Stokes equations with Hodge Laplacian (2.9).

Theorem 5.1.1 Let $u$ be a solution of (5.1) and $v$ be Killing. Then

$$
\frac{d}{dt} \langle u, v \rangle = 0.
$$
The proof of theorem 5.1.1 is straightforward, follows lemma 2.3.4 and the properties of the diffusion operator $Lu$. Consequently,

$$
\frac{d}{dt} (u,v) = \langle u_t, v \rangle = -\langle \nabla u, v \rangle + \mu \langle Lu, v \rangle - \langle \nabla (p), v \rangle = 0 .
$$

Theorem 5.1.2 Let $u$ be a solution of the Navier–Stokes equations with Hodge Laplacian (2.9), and let $\nu$ be harmonic. Then

$$
\frac{d}{dt} (u, v) = 0 .
$$

Theorem 5.1.2 implies theorem 5.1.1 by applying lemma 2.3.5.

We now suppose $(u, p)$ is a solution of Navier–Stokes equations that can be decomposed as $u = u^K + u^\perp$, where $u^K$ is Killing and $u^\perp$ is orthogonal to the Killing fields. The whole dynamics of the solution $u = u^K + u^\perp$ thus occurs in the component $u^\perp$. Then write $p = p_K + p:\perp$ where $p_K = \frac{1}{2} g(u^K, u^K)$. The following system for $u^\perp$ is valid from (5.1):

$$
\begin{align*}
{u^\perp}_t + \nabla_{u^\perp} u^K + \nabla_{u^K} u^\perp + \nabla_{u^\perp} u^\perp - \mu Lu^\perp + \nabla (p) &= 0 \\
\text{div}(u^\perp) &= 0 .
\end{align*}
$$

(5.2)

Theorem 5.1.3 Let $u^\perp$ be a solution of (5.2). Then

$$
\|u^\perp\|^2 \leq Ce^{-\mu \alpha_{K}t} .
$$

According to lemma 2.3.3 and formula (2.10), the following conditions hold

$$
\int_M g(\nabla_{u^\perp} u^K, u^\perp) \omega_M = \int_M g(\nabla_{u^K} u^\perp, u^\perp) \omega_M = \int_M g(\nabla_{u^\perp} u^\perp, u^\perp) \omega_M = 0 .
$$

Then lemma 2.3.6 and lemma 2.3.1 complete the proof. In particular, the solutions of the stationary problem

$$
\begin{align*}
\nabla u - \mu Lu + \nabla (p) &= 0 \\
\text{div}(u) &= 0
\end{align*}
$$

are precisely the Killing vector fields.

5.1.2 Linearized Navier–Stokes equations

We apply a linear approximation for a nonlinear system which is called linearization. This method makes analysis easier and designs the system in which the results are close to reality. For example, under this process we can take the gradient of a nonlinear function with respect to all elements and set up a linear idea at that point. The first step in linearization is to identify nonlinear components and write nonlinear differential equations. When we linearize a nonlinear differential equation we in fact linearize a system for small signal inputs around a steady state.
Consequently, for solving Navier–Stokes equations one has to linearize this system as follows:

\[
\begin{align*}
    u_t + \nabla u v + \nabla v u - \mu L u + \text{grad}(p) &= 0 \\
    \text{div}(u) &= 0.
\end{align*}
\]  

(5.3)

The variational formulation can then be written as

\[
\frac{d}{dt} \langle u, w \rangle = -\langle \nabla u v, w \rangle - \langle \nabla v u, w \rangle + \mu \langle L u, w \rangle - \langle \text{grad}(p), w \rangle.
\]

The performance of this method has been validated to implement results and analytical solutions in several cases by considering \( w \) as a Killing vector field. By linearizing the Navier–Stokes system, the significant result would be

\[
\frac{d}{dt} \langle u, w \rangle = 0
\]

for any vector field \( u \) which is a solution of (5.3) and considering \( w \) as a specific vector field in the role of Killing.

Alternatively, we can decompose the solution of the Navier–Stokes system as \( u = u^K + u^\perp \), where \( u^K \) is Killing and \( u^\perp \) is orthogonal to the Killing field. Similarly,

\[
p = p_K + p_\perp = g(u, v) + p_\perp,
\]

where \( v = v^K + v^\perp \) is any vector field and denote \( f = -\nabla u^K v^\perp - \nabla v^\perp u^K \). Then we can write the linear system (5.3) for \( u^\perp \) as follows:

\[
\begin{align*}
    u^\perp_t + \nabla v^K u^\perp + \nabla u^\perp v^K + \nabla v^\perp u^\perp + \nabla v^K u^\perp - \mu L u^\perp + \text{grad}(p_\perp) &= f \\
    \text{div}(u^\perp) &= 0.
\end{align*}
\]  

(5.4)

Theorem 5.1.4 Let \( u^\perp \) be a solution of (5.4). Then

\[
\frac{d}{dt} \| u^\perp \|^2 \leq -\mu \alpha_K \| u^\perp \|^2 + 2 \langle f, u^\perp \rangle - 2 \langle \nabla u^\perp v^\perp, u^\perp \rangle.
\]

The proof of theorem 5.1.4 follows easily since by straightforwardly considering \( v^K \) as a Killing vector field, from the main following formula

\[
\frac{1}{2} \frac{d}{dt} \langle u^\perp, u^\perp \rangle = \langle f, u^\perp \rangle - \langle \nabla v^K u^\perp, u^\perp \rangle - \langle \nabla v^\perp u^K, u^\perp \rangle - \langle \nabla u^\perp v^\perp, u^\perp \rangle - \langle \nabla v^\perp u^\perp, u^\perp \rangle + \mu \langle L u^\perp, u^\perp \rangle - \langle \text{grad}(p), u^\perp \rangle
\]

we get

\[
\frac{1}{2} \frac{d}{dt} \langle u^\perp, u^\perp \rangle = \langle f, u^\perp \rangle - \langle \nabla u^\perp v^\perp, u^\perp \rangle + \mu \langle L u^\perp, u^\perp \rangle.
\]

Then the result will be obtained by combining theorem 2.3.6 with lemma 2.3.1.

### 5.1.3 New solutions from the old

In this section, we first state a general properties of Killing vector fields, and then produce new solutions with the bracket of Killing vector fields for the linearized system of Navier–Stokes equations.
Theorem 5.1.5 Let \( (u, p) \) be a solution of (5.3) where we suppose that \( v \) is Killing. Then

\[
(\hat{u}, \hat{p}) = \left( [u, v], -g(\nabla p, v) \right)
\]

is also a solution of (5.3).

It would be easy to confirm that \( \nabla \cdot \hat{u} = \nabla \cdot [u, v] = 0 \). The following equation applies to \( \hat{u} \):

\[
\]

The following claims can easily be checked and verified.

Claim 1. \( [\nabla(p), v] = \nabla(\hat{p}) \).

Claim 2. \( \nabla_v [u, v] = [\nabla_v u, v] \).

Claim 3. \( \nabla_{[u,v]} v = [\nabla_u v, v] \).

Claim 4. \( \Delta_B [u, v] = [\Delta_B u, v] \).

Claim 5. \( \text{Ri}[u, v] = [\text{Ri}(u), v] \).

5.1.4 Two-dimensional case

Since one of the main factors for the study of flows comes from atmospheric models, the case can be more specifically focused on the Coriolis effect in the two-dimensional space as

\[
\begin{align*}
u_t + \nabla u - \mu \Delta_B u - \mu \kappa u + \nabla p &= 0 \\
- \Delta p - \text{tr}((\nabla u)^2) - \kappa g(u, u) + 2 \mu g(\nabla(\kappa), u) &= 0 \\
\text{div}(u) &= 0.
\end{align*}
\]

Due to the importance of vorticity in most fluid issues, we will analyze this in our context. An equation for the vorticity in incompressible flow in the two-dimensional space is obtained by applying the \( \text{div} \circ K \) operation to the Navier–Stokes equation by applying the identity \( \text{rot} \circ \nabla = 0 \), so the pressure term vanishes. For the inertia term \( \text{rot}(\nabla u) \), this would be easy to check

\[
\varepsilon_i^{lj} u_i^l u_j^l = \varepsilon_i^{lj} u_l^l u_i^j = \zeta \text{div}(u)
\]

The result is therefore

\[
\zeta_t - \mu \Delta \zeta + g(\nabla(\zeta), u) - 2 \mu g(\nabla(\kappa), Ku) - 2 \mu \kappa \zeta = 0.
\]

In the above equation, \( \Delta \zeta \) represents the action of velocity variations on the vorticity and \( g(\nabla(\zeta), u) \) describes the convective transport of vorticity. The advantage of using this form of Navier–Stokes equations is that it reduces the number of independent variables and then reduces the necessary memory space. In this form, we do not need to determine the boundaries for our system. However, the value for considering these two advantages increases the order of the differential equation from two to four.
Theorem 5.1.6 Let $\zeta$ be the solution of (5.6) on the sphere; then
\[ \frac{d}{dt} \|\zeta\|^2 \leq 0. \]

Our proof of theorem 5.1.6 is directly completed by applying the inequality
\[ \lambda_1 \int_M f^2 \omega_M \leq \int_M g(\nabla(f), \nabla(f)) \omega_M \]
because $\int_M \zeta \omega_M = 0$ and on the sphere $\lambda_1 = 2\kappa$, where $\lambda_1$ is the first positive eigenvalue of $-\Delta$.

Let us again use the decomposition $u = u^K + u^\perp$ for the solution of (2.5) and let $\zeta = \zeta^K + \zeta^\perp$ be the corresponding decomposition for the vorticity. On the sphere we can check
\[ \langle \zeta^K, \zeta^\perp \rangle = 0. \]

5.1.5 Coriolis

The Navier–Stokes system on the unit sphere by adding the Coriolis effect can be considered as
\[
\begin{align*}
\frac{du}{dt} + \nabla_u u - \mu Lu + a Ku + \nabla(p) &= 0 \\
-\Delta p - \text{tr}(\nabla u^2) - g(u, u) - \text{div}(a Ku) &= 0 \\
\text{div}(u) &= 0. 
\end{align*}
\]
(5.7)

Since $g(Ku, u) = 0$ the Coriolis term has no effect on the norm: we still have
\[ \frac{d}{dt} \|u\|^2 = -\mu \int_M g(S_u, S_u) \omega_M. \]

However, not all Killing fields are now solutions. Let $u$ be Killing; if the Killing vector field $u$ is a solution to (5.7), then we should have
\[ \text{rot}(a Ku) = -a \text{div}(u) - g(\nabla(a), u) = -g(\nabla(a), u) = 0. \]

The only Killing vector field in spherical coordinates which satisfy the above condition is $u = c \partial_\theta$, where $c$ is constant. We now consider $u$ in the formula (5.7) as our Killing vector field, so
\[ \nabla(p) = -c^2 \nabla_\theta \partial_\theta - a K(c \partial_\theta) \]
and the corresponding pressure is then
\[ p_K = \frac{1}{2} (c^2 \sin(\varphi)^2 + c \omega \cos(2\varphi)). \]

Let us suppose $(u^K, p_K)$ as a solution for (5.7), which is given as the above formula in spherical coordinates. Then we can try to find $u = u^K + \hat{u}$ and $p = p_K + \hat{p}$ for this form of solutions.

Theorem 5.1.7 Let $u = u^K + \hat{u}$, $p = p_K + \hat{p}$ be a solution to (5.7). Then
\[ \frac{d}{dt} \|\hat{u}\|^2 \leq 0. \]
For the proof of theorem 5.1.7, the variational formulation for (5.7) can be written as

\[
\frac{1}{2} \frac{d}{dt} \| \hat{u} \|^2 = - \int_M g(\nabla_{\hat{u}} K, \hat{u}) \omega_M - \int_M g(\nabla_{\hat{u}} \hat{u}, \hat{u}) \omega_M - \int_M g(\nabla \hat{u}, \hat{u}) \omega_M \\
+ \mu \int_M g(L \hat{u}, \hat{u}) \omega_M - \int_M \hat{a} g(K \hat{u}, \hat{u}) \omega_M - \int_M g(\text{grad}(\hat{p}), \hat{u}) \omega_M.
\]

By then applying lemma 2.3.3 and the formula

\[
\int_M g(\nabla_{\hat{u}} u, v) \omega_M = \int_M g(\nabla_{\hat{u}} v, u) \omega_M = 0,
\]

if either (i) \( u \) is Killing and \( \text{div}(v) = 0 \) or (ii) \( v \) is Killing and \( \text{div}(u) = 0 \), the proof is completed.
5.2 SUMMARY OF PAPER II

The equations which are defined for Killing and conformal Killing vector fields are overdetermined systems of PDE and difficult to solve numerically. So we try to solve them by the eigenvalue problem.

5.2.1 Killing, conformal Killing vector fields and Eigenvalue problem

We defined (2.1) two operations $S$ and $C$ for every $u$ field on Riemannian manifold $M$ as follows

$$(Su)^{kj} = g^{kj}u^{ij} + g^{kj}u^{kj}$$

$$(Cu)^{kj} = (Su)^{kj} - \frac{2}{n}\text{div}(u)g^{kj}$$

As a PDE system $Su = 0$ is a system of $\frac{1}{2}n(n + 1)$ linear first-order equations with $n$ unknown functions.

The $Cu = 0$ system is a system of $\frac{1}{2}n(n + 1)$ equations with $n$ unknowns, but the dimension $n = 2$ is now a particular case. In $n = 2$, only two independent equations exist, and the resulting system is easily seen as elliptical. Hence, locally the space of conformal Killing fields is infinite dimensional. When $n > 2$ the system is finite in type so in this case the solution space is finite dimensional even locally.

Eigenvalue problem and Coercive

In this section we will always suppose $M$ is a compact Riemannian manifold. Let $V$ be a real Hilbert space and let $a : V \times V \to \mathbb{R}$ be a continuous and symmetric bilinear map. Let $H$ be another Hilbert space such that $V \subset H$ with compact and dense injection. Let us consider the following eigenvalue problem:

find $\lambda$ and $u \neq 0$ such that

$$a(u, v) = \lambda \langle u, v \rangle_H$$

for all $v \in V$.

Due to symmetry the eigenvalues are real. We say that $a$ is coercive if there are two constants $\alpha > 0$ and $\mu \in \mathbb{R}$ such that

$$a(v, v) + \mu\|v\|_H^2 \geq \alpha\|v\|_V^2 \quad \forall v \in V.$$ 

If $a$ is symmetric, continuous and coercive, then there are real numbers $\lambda_k$ and elements $u_k \in V$ such that

$$a(u_k, v) = \lambda_k \langle u_k, v \rangle_H, \quad \forall v \in V,$$

where $-\mu < \lambda_1 \leq \lambda_2 \leq \ldots$ and $\lambda_k \to \infty$ when $k \to \infty$. Moreover all eigenspaces are finite dimensional and they are orthogonal to each other with respect to the inner product of $H$.

As a straightforward application of section 2.1.1., we introduce the following bilinear maps:

$$a_K : H^1(M) \times H^1(M) \to \mathbb{R}, \quad a_K(u, v) = \frac{1}{2} \int_M g(S_u, S_v)\omega_M$$

$$a_C : H^1(M) \times H^1(M) \to \mathbb{R}, \quad a_C(u, v) = \frac{1}{2} \int_M g(C_u, C_v)\omega_M.$$
Then we can formulate the following eigenvalue problems:

(K) Find \( u \in H^1(M) \) and \( \lambda \) such that

\[
a_K(u,v) = \lambda \int_M g(u,v) \omega_M
\]

for all \( v \in H^1(M) \).

(CK) Find \( u \in H^1(M) \) and \( \lambda \) such that

\[
a_C(u,v) = \lambda \int_M g(u,v) \omega_M
\]

for all \( v \in H^1(M) \).

Thus, obviously \( a_K(u,u) \geq 0 \) and \( a_C(u,u) \geq 0 \) for all \( u \), and \( a_K(u,u) = 0 \) (resp. \( a_C(u,u) = 0 \)) only if \( u \) is Killing (resp. conformally Killing) (lemma 2.2.1 and lemma 2.2.3). Therefore, the eigenspace of a zero eigenvalue is the space of the Killing fields (resp. conformally Killing fields). It is clear that \( a_K \) and \( a_C \) are symmetric and continuous so that \( \lambda \), in particular, must be real. We must then show that the maps \( a_K \) and \( a_C \) are coercive.

**Theorem 5.2.1** The maps \( a_K \) and \( a_C \) are coercive, if \( M \) has no boundary.

Our proof of theorem 5.2.1 is directly obtained by applying formula (2.4) and (2.8) as follows:

\[
div(S_u v) = \frac{1}{2} g(S_u S_u) + g(L_u v)
\]

\[
div(C_u v) = \frac{1}{2} g(C_u C_u) + g(L_C u, v)
\]

Then continue by applying lemma 2.1.1 and lemma 2.1.2, we have:

\[
L_K u = \Delta_B u + \text{grad}(\text{div}(u)) + \text{Ri}(u)
\]

\[
L_C u = \Delta_B u + (1 - \frac{2}{n}) \text{grad}(\text{div}(u)) + \text{Ri}(u).
\]

Therefore, on the unbounded manifolds

\[
a_K(u,u) = \int_M \left( g(\nabla u, \nabla u) + \text{div}(u)^2 - \text{Ri}(u, u) \right) \omega_M
\]

\[
\geq \int_M \left( g(\nabla u, \nabla u) + \left(1 - \frac{2}{n}\right) \text{div}(u)^2 - \text{Ri}(u, u) \right) \omega_M
\]

\[
= a_C(u,u) \geq \int_M \left( g(\nabla u, \nabla u) - \text{Ri}(u, u) \right) \omega_M.
\]

We can define \( \mu = \max_{p \in M} \|\text{Ri}\| \). Since \( M \) is compact, \( \mu \) is finite. Hence,

\[
a_K(u,u) \geq a_C(u,u) \geq \int_M \left( g(\nabla u, \nabla u) - \mu g(u,u) \right) \omega_M \geq \alpha \|u\|^2_{H^1} - (\mu + \alpha) \|u\|^2_{L^2}
\]

if \( 0 < \alpha \leq 1 \).

**Remark 5.2.1** If \( u \) is Killing then

\[
\int_M g(\nabla u, \nabla u) \omega_M = \int_M \text{Ri}(u, u) \omega_M
\]

and if \( u \) is conformally Killing then

\[
\int_M \left( g(\nabla u, \nabla u) + (1 - \frac{2}{n}) \text{div}(u)^2 \right) \omega_M = \int_M \text{Ri}(u, u) \omega_M.
\]
Therefore, our eigenvalue problems are now well defined as follows:

\[ a_K(u,v) = \int_M \left( g(\nabla u, \nabla v) + \operatorname{tr}(\nabla u \nabla v) \right) \omega_M \]

\[ a_C(u,v) = \int_M \left( g(\nabla u, \nabla v) + \operatorname{tr}(\nabla u \nabla v) - \frac{2}{n} \operatorname{div}(u) \operatorname{div}(v) \right) \omega_M \, . \]

Let \( p \in \partial M \) and let \( \{ \tau_1, \ldots, \tau_{n-1} \} \) be a basis of \( T_p \partial M \) and let \( v \) be the outer unit normal vector. By using operators \( L \) and \( L_C \) we can write the eigenvalue problems as the following formula:

(K0) Find \( u \) and \( \lambda \) such that

\[
\begin{cases}
- L_K u = \lambda u \\
g(\nabla \nu u, \tau_k) + g(\nabla \tau_k u, v) = 0, \\
g(\nabla \nu v, v) = 0.
\end{cases}
\]

(CK0) Find \( u \) and \( \lambda \) such that

\[
\begin{cases}
- L_C u = \lambda u \\
g(\nabla \nu u, \tau_k) + g(\nabla \tau_k u, v) - \frac{2}{n} \operatorname{div}(u) \operatorname{div}(v) = 0, \\
2g(\nabla \nu v, v) - \frac{2}{n} \operatorname{div}(u) g(\nu, v) = 0.
\end{cases}
\]

Remark 5.2.2 If \( u \) is Killing (resp. conformally Killing) then it satisfies the boundary conditions of the problem (K0) (resp. problem (CK0)).

5.2.2 Numerical results

We will solve problems (K) and (CK) in three cases: Enneper’s surface, torus and the Klein bottle. In all cases, we used FREEFEM++ to visualize the computed solutions.

Two-dimensional case

We assume \( u \) is a Killing vector field. As in definition 2.2.1, the Killing equations are

\[
\begin{align*}
& g^{11} u_1^1 + g^{12} u_2^1 = 0, \\
& g^{11} u_1^2 + g^{12} u_2^2 + g^{12} u_1^1 + g^{22} u_2^1 = 0, \\
& g^{12} u_1^2 + g^{22} u_2^2 = 0.
\end{align*}
\]

Let \( v = Ku \); As in definition 2.2.2, the conformal Killing equations are

\[
\begin{align*}
& g^{11} v_1^1 + g^{22} v_2^1 = 0, \\
& g^{11} v_1^2 + 2g^{12} v_2^1 - g^{11} v_2^2 = 0.
\end{align*}
\]

This gives us the lemma:

Lemma 5.2.1 Let \( u \) be a Killing field. Then \( Ku \) is a conformal Killing field.

Definition 5.2.2 A coordinate system of a two-dimensional Riemannian manifold is isothermal, if the metric takes the form

\[ g = e^{\lambda(x)} (dx_1 \otimes dx_1 + dx_2 \otimes dx_2) \]

for some function \( \lambda \).
If the parametrization is then isothermal, we find that

\[
Su = 0 \iff \begin{cases}
    u^1 \lambda_{1,1} + u^2 \lambda_{2,1} + 2u^2 = 0 \\
    u^2_1 + u^2_2 = 0 \\
    u^1_1 - u^2_2 = 0.
\end{cases}
\]

(5.8)

Three-dimensional case

We introduced the parametrization of a surface of revolution in \( \mathbb{R}^3 \) (section 3.1.1). Here we will present the Killing and conformal Killing vector fields in regard to that.

Lemma 5.2.2. On the surfaces of revolution, vector fields \( b \partial_{x_2} \) where \( b \) is constant are Killing fields. There are no other Killing fields unless the profile curve has a constant curvature.

In the proof of lemma 5.2.2, the Killing equations (definition 2.2.1) for the surfaces of revolution leads to

\[
|c'|^2 u^1_1 + \langle c', c'' \rangle u^1 = 0 \\
|c'|^2 u^2_2 + c^2_2 u^2_1 = 0 \\
c_1 u^1_2 + c'_1 u^1 = 0.
\]

Clearly the fields \( u = b \partial_{x_2} \) are solutions and there can be no other Killing fields. Therefore, we have a conformal Killing field at the surface of the revolution by lemma 5.2.1:

\[
v = K \partial_{x_2} = g^{11} \epsilon_{12} \partial_{x_1}.
\]

(5.9)

Enneper’s surface

We introduced Enneper’s surface, section 3.1.1, in \( \mathbb{R}^3 \). Enneper’s surface is isothermal (definition 5.2.2) when

\[
\lambda = 2 \ln (1 + |x|^2).
\]

Lemma 5.2.3. Vector fields \( u = -bx_2 \partial_{x_1} + bx_1 \partial_{x_2} \) where \( b \) is a constant, then there are Killing fields on Enneper’s surface.

By applying system (5.8), we have

\[
\begin{cases}
    2x_1 u^1 + 2x_2 u^2 + u^2_2 (1 + |x|^2) = 0 \\
    u^2_1 + u^2_2 = 0 \\
    u^1_1 - u^2_2 = 0
\end{cases}
\]

and using the program Maple, the above system is equivalent to

\[
\begin{cases}
x_1 u^1 + x_2 u^2 = 0 \\
    u^2_1 = 0 \\
    x_1 u^2_1 - u^2 = 0.
\end{cases}
\]
Torus

In chapter 3.1.1 we introduced a torus in $\mathbb{R}^3$. The flat torus is isothermal with $\lambda = 0$ so that in this case the Killing fields are $u = b_1 \partial_{x_1} + b_2 \partial_{x_2}$.

Let us then consider the "standard" torus, with its Riemannian metric defined by the embedding in $\mathbb{R}^3$. This is a surface of revolution and as a profile curve we can choose

$$c(x_1) = (2 + \cos(x_1), \sin(x_1)) .$$

The corresponding metric is given by

$$g = dx_1 \otimes dx_1 + (2 + \cos(x_1))^2 dx_2 \otimes dx_2 .$$

By lemma 5.2.2, $u = \partial_{x_2}$ is a Killing field and formula (5.9) indicates that

$$v = Ku = K \partial_{x_2} = (2 + \cos(x_1)) \partial_{x_1}$$

is a conformal Killing field.

Klein bottle

Finally, let us consider the Klein bottle so that our method works on non-oriented surfaces as well. By applying metric (3.1), the Killing equations are now

$$u^1_{,1} = 0$$

$$4u^1_{,2} + a u^2_{,1} = 0$$

$$a u^2_{,2} - b u^1 = 0 ,$$

where $a = 3 \cos^2(x_1) + 16 \cos(x_1) + 17$ and $b = \sin(x_1)(3 \cos(x_1) + 8)$. It is easy to confirm that the solution is $u = \partial_{x_2}$ . Then, by lemma 5.2.1, the vector field

$$v = Ku = -\frac{\sqrt{3 \cos(x_1)^2 + 16 \cos(x_1) + 17}}{2} \partial_{x_1}$$

is a conformal Killing field.
5.3 SUMMARY OF PAPER III

We will analyze and present examples of several classes of Riemannian manifolds which are defined by imposing a certain condition on the Ricci tensor. The following cases are considered: The Ricci recurrent, Cotton, quasi Einstein and pseudo Ricci symmetric conditions. Such conditions can be seen as PDE systems that are overdetermined and unknown as Riemannian metric components.

5.3.1 Some properties and relationships between various classes

Below we consider several classes of Riemannian manifolds. Such classes are categorized as requiring the Ricci tensor to satisfy some $P$ condition. In this case, we can also say that the manifold or Riemannian metric is of the $P$ type.

Ricci recurrent

As in formula (4.3), we can identify the associated one-form $\beta$ in RR manifolds as follows:

$$\beta = \frac{R^{ij} R_{ij}}{|R|^2} = \frac{1}{2} (\nabla \ln(|R|^2))_\ell .$$

Lemma 5.3.1 Let $NR_i = \frac{R_i}{|R|^2}$. Then $R_i$ is RR if and only if $NR_i$ is parallel.

Our proof of lemma 5.3.1 is directly obtained by taking the covariant derivative of $NR_i$.

The following theorem aids in characterizing the Ricci tensor.

Theorem 5.3.1 Suppose that $R_i$ is recurrent. Then it has a double eigenvalue $\frac{sc}{2}$ and an eigenvalue zero of multiplicity $n - 2$. Moreover,

$$\beta = \nabla \ln(sc) , \quad sc^2 = 2 |R|^2 \quad \text{and} \quad R_i \beta = \frac{sc}{2} \beta .$$

By characterizing RR in an algebraic way, one can show that

$$R_i^j R_{i}^j = \frac{1}{2} sc R_{i}^k$$

and then by considering $\lambda_j$ as the eigenvalues of $R_i$, we get

$$\lambda_k^2 = \frac{1}{2} (\lambda_1 + \cdots + \lambda_n) \lambda_k .$$

As the scalar curvature is not zero, the above equation holds only when $\lambda_1 = \lambda_2 \neq 0$ and $\lambda_3 = \cdots = \lambda_n = 0$; by multiplying formula (5.10) by $g^{ij}$ the proof is completed. One application of the formulation above may be as follows: On one hand, we can easily confirm that if the Ricci tensor satisfies the RR condition, the Schouten tensor is also recurrent, as in following formula:

$$S_{ijk} = \frac{sc_k}{sc} \left( R_i - \frac{1}{2(n-1)} sc g_{ij} \right) = \frac{sc_k}{sc} S_{ij} .$$

On the other hand, we can note from the previous theorem that $R_i$ has a $v$ eigenvector such that $R_i v = 0$ and $g(\text{grad}(sc), v) = 0$. Therefore, we get

$$C_{ijk} v^j = S_{ijk} v^j - S_{ikj} v^j = -\frac{1}{2(n-1)} sc_k v_i \neq 0 .$$
Pseudo Ricci symmetric

By multiplying (4.7) by $g^{ij}$, the associated one-form $\alpha$ in PRS condition can be identified as:

$$\alpha = \frac{1}{4} \nabla \ln(|Ri|^2).$$

Furthermore, if the scalar curvature is not zero then

$$\alpha = \frac{1}{2} \nabla \ln(sc) \quad \text{and} \quad sc^2 = c |Ri|^2$$

for some constant $c$.

We can easily check, by citing formula (4.8), that if the scalar curvature in the PRS is constant, it must be zero. However, whether the case $sc = 0$ occurs seems to be an open question. If we do not assume that the metric is positive definite then it is easy to construct examples with $sc = 0$. We could not, however, find an example with a positive definite metric.

Remark 5.3.1 We can easily verify:

- The Ricci tensor cannot be both RR and PRS.
  
  In both cases, it is easy to confirm that the non-zero scalar curvature implies that

  $$sc_{\ell} = 2 \alpha_{\ell}sc$$
  $$sc_{\ell} = \beta_{\ell}sc.$$  

  Therefore $\beta = 2\alpha$. Indeed, by applying formulas (4.7) and (4.1) the result will be $|\alpha|^2Ri = 0$, which is impossible.

- If $Ri$ is parallel then the PRS condition cannot be satisfied.
  
  If $Ri$ is parallel, then multiplying (4.7) by $a^\ell$ gives $2|\alpha|^2Ri = 0$, which is impossible.

Quasi Einstein

We analyzed the quasi Einstein structure in section 4.1.4 when introducing the tensor $T = Ri_{ij} - a g_{ij}$. We could say

(i) $M$ is an Einstein manifold, if $T = 0$ and $a = sc/n$.

(ii) $M$ is a quasi Einstein manifold, if the matrix rank of $T$ is one.

In this way, we see that in the analysis of the existence of a quasi Einstein structure the one-form $\omega$ and the function $b$ are quite irrelevant.

Another way to characterize the QE case, which is useful when considering examples, is that $Ri$ has a simple eigenvalue $sc - (n - 1)a$ corresponding to the eigenvector $\omega$. All vectors orthogonal to $\omega$ are eigenvectors with eigenvalue $a$ whose multiplicity is thus $n - 1$.

Theorem 5.3.2 If the Ricci tensor is recurrent and $n = 3$, then it is automatically quasi Einstein. If $n > 3$, the Ricci tensor cannot be both recurrent and quasi Einstein.
In the proof of theorem 5.3.2, we can conclude from theorem 5.3.1 that the eigenvalue structure of $R_i$ can satisfy both RR and QE conditions only if $n = 3$. In this case, if $\omega$ is matched to the eigenvector corresponding to the zero eigenvalues, then the Ricci tensor may be written as follows:

$$R_{ij} = \frac{sc}{2} \left( g_{ij} - \frac{\omega_i \omega_j}{|\omega|^2} \right).$$

**Theorem 5.3.3** Let us suppose that both PRS and QE conditions are satisfied and let $\alpha$ be the associated one-form. If $a \neq 0$ we have

$$R_{ij} = \frac{sc}{n-1} \left( g_{ij} - \frac{\alpha_i \alpha_j}{|\alpha|^2} \right).$$

If $a = 0$ then

$$g(\omega, \alpha) = 0 \quad , \quad R_{ij} = \frac{\omega_i \omega_j}{|\omega|^2} \quad \text{and} \quad \nabla_\alpha \omega = |\alpha|^2 \omega.$$  

The condition

$$aa_j + b \frac{g(\omega, \alpha)}{|\omega|^2} \omega_j = 0 \quad (5.11)$$

occurs when both cases PRS and QE are satisfied. Two cases occur

(i) If $a \neq 0$ and $g(\omega, \alpha) \neq 0$, then $\alpha$ and $\omega$ are linearly dependent and we may choose $\omega = \alpha$, which gives

$$a = \frac{sc}{n-1} \quad \text{and} \quad b = -a.$$  

(ii) When $a = 0$ the first two statements are obvious. To obtain the third we take the covariant derivative of the formula $R_{ij} = \frac{\omega_i \omega_j}{|\omega|^2}$, and then multiply it by $\alpha^i$. By applying formula (4.7) the proof is completed.

Let $R_i$ satisfy the PRS with non-zero scalar curvature, then the Cotton tensor is

$$C_{ijk} = \frac{sc}{2} \left( R_{ij} - \frac{sc}{n-1} g_{ij} \right) - \frac{sc}{2} \left( R_{ik} - \frac{sc}{n-1} g_{ik} \right).$$

Regarding theorem 5.3.3, $R_i$ also satisfies QE with $a \neq 0$. It seems that the above type of Cotton tensor can only be zero if the matrix rank of $R_{ij} - \frac{sc}{n-1} g_{ij}$ is one, that is the exact condition of QE.

We can construct QE in another way that we now describe. Let $\hat{g} = \exp(2\lambda) g$ be as a metric that is conformally equivalent to the metric $g$ and $\hat{R}_i$ a Ricci tensor associated to $\hat{g}$ defined as

$$\hat{R}_{ij} = R_{ij} - (n - 2)(\lambda,_{ij} - \lambda,_{ji}) - (\Delta \lambda + (n - 2)|\nabla \lambda|^2) g_{ij}.$$  

We can easily check if we could find a metric $g$ and a function $\lambda$ such that $R_{ij} = (n - 2)\lambda,_{ij}$ then $\hat{R}_i$ is quasi Einstein:

$$\hat{R}_{ij} = -(\Delta \lambda + (n - 2)|\nabla \lambda|^2) \exp(-2\lambda) \hat{g}_{ij} + (n - 2)\lambda,_{ij}.$$  

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Remember that, if \( R_i = \nabla \nabla \lambda = 0 \) the equations are trivially satisfied. In [3], the presence of the solutions of the PDE system \( R_i = T \) where \( T \) is a given symmetric tensor is analyzed. Let us formulate the Bianchi identity (A.4) and its modified form, \( B \) and \( B' \), for tensor \( T \) as follows:

\[
B(T) = 2 \text{div}(T) - \nabla \text{tr}(T) = 2 g^{ij} T_{ik;j} - g^{ij} T_{ij;k}
\]

\[
B'(T) = 2 g^{ij} T_{ik;j} - g^{ij} T_{ij;k}.
\]

Furthermore, if the \( R_i = T \) system has solutions

\[ B(T) = 0 \]

and \( B'(R_i) = B'(T) \), this brings us to a third-order system with an unknown \( \lambda \) as follows:

\[
\begin{aligned}
R_i &= (n - 2) \lambda_{ij} \\
B'(R_i) &= (n - 2) B'(\lambda_{ij}) \\
B(\lambda_{ij}) &= 0.
\end{aligned}
\] (5.12)

### 5.3.2 Examples based on PDE system

In this section, we solve our PDE systems that named them \( P \) conditions. Therefore, more precisely, we state the following assumptions:

- Because our analysis is local, only the situation in a single coordinate system will always be considered.
- All unknown functions will be defined as functions of one variable only, so our PDE systems can be reduced to ODE systems and thus finding solutions is simpler.
- We are not interested in cases where some unknown functions vanish, or the differential of the scalar curvature vanishes, or that \( R_i = 0 \).

In this way, we need to use the algorithm \( \text{rifsimp} \) that automatically governs the equations which are actually algebraically independent. This algorithm tries to express highest ranking derivatives in terms of lower ranking derivatives. Let \( f = (f_1, \ldots, f_k) \) be the unknown functions with \( x = (x^1, \ldots, x^n) \) as independent variables. Let \( \alpha \) be some multi-indices. Then the first part of output is as follows:

\[
\partial^{\alpha j} f_j = F_j(x, f, \ldots), \quad 1 \leq j \leq m .
\] (5.13)

In the arguments of \( F_j \) there are only lower-ranking derivatives than \( \partial^{\alpha j} f_j \). \( \text{rifsimp} \) also tries to eliminate the components \( f_j \) from equations as far as possible. If the problems are nonlinear, then all relevant system information can’t always be expressed as in (5.13). In such instances, there are also additional equations in \( \text{rifsimp} \), which are called \( \text{constraints} \), of the form

\[
H_\ell(x, f, \ldots) = 0, \quad 1 \leq \ell \leq s ,
\]

where the highest-ranking derivatives of the arguments of \( H_\ell \) are present nonlinearly. Sometimes we have to make some decisions when computing \( F_j \) and \( H_\ell \), if certain expressions are zero or not. But of course, if one term is zero, solutions that are not part of the general solution can be made. We often used the \text{dsolve} command in Maple when solving the equations.
Four-dimensional case

Let us consider a Riemannian metric $g$ on $\mathbb{R}^4$ given by

$$g = f_1(x^1)(dx^1)^2 + f_2(x^1)(dx^2)^2 + f_3(x^1)(dx^3)^2 + f_4(x^1)(dx^4)^2.$$  \hspace{1cm} (5.14)

- **RR condition:**
  In this condition, we set two of the functions $f_2$, $f_3$ and $f_4$ as constants. For instance, the choice is $f_3 = f_4 = 1$. As in theorem 5.3.1, we have $\beta = \nabla \ln(sc)$ where

$$sc = \frac{-2f_1f_2f_3'' + f_1(f_2')^2 + f_2f_1f_2'}{2f_1f_2^2}$$

In particular, the question reduces to the two-dimensional case and, naturally, any metric in two-dimensions is RR.

- **PRS condition:**
  The system is decomposed into eight components in which the most general condition: any metric in two-dimensions is RR. In particular, the case we have initially six equations, but rifsimp gives only the following two equations:

$$f_2''' = F_2(f_1, f_2, f_3, f_4)$$
$$f_3''' = F_3(f_1, f_2, f_3, f_4).$$

By arbitrarily choosing $f_1$ and $f_4$, we have a standard ODE system for $f_2$ and $f_3$. 

This results in:

$$f_1 = c_1c_2c_4m^2 \exp(c_3m+1)f^{(m+1)/f^{(m+1)}}$$
$$f_2 = c_2f^m$$
$$f_4 = c_4 \exp(c_3m+1)f^{m/(m+1)}.$$
• First QE condition:
First, we set the minors of matrix \( T = \text{Ri} - a \). Then, Maple will show the following results:

\[
a = \frac{f_1 f_2 f_3 f_4^2 + f_1 f_2 f_4 f_3^2 f_4' - f_1 f_3 f_4 (f_2')^2 - f_2 f_3 f_4 f_1 f_2' - 2 f_1 f_2 f_3 f_2' f_4}{4 f_1^2 f_2^2 f_3 f_4}.
\]

This yields \( b = sc - 4a \), so

\[
b = B(f_1, f_2, f_3, f_4).
\]

Here the expression of \( B \) is so large that we do not give it explicitly. After choosing \( a \) and \( b \), we do not have a simple PDE to solve. There are three families of solutions and as usual, we can provide the most general. First rifsimp gives:

\[
f''_2 = \frac{2 f_1 f_2 f_3 f_4 f''_4 - f_1 f_2^2 f_3 (f_4')^2 + f_1 f_2^2 f_4 f_3 f'_4 - f_1 f_2 f_3 f_4 f'_4 f''_4 + f_1 f_3 f_4 (f''_2)^2}{2 f_1 f_2 f_3 f_4^2}
\]

\[
+ \frac{-f_1^2 f_3 f_4 f_1 f_4' + f_2 f_3 f_4^2 f_1 f'_2}{2 f_1 f_2 f_3 f_4^2}.
\]

\[
f''_3 = \frac{2 f_1 f_2 f_3^2 f_4 f''_4 - f_1 f_2 f_3 (f'_4)^2 + f_1 f_2 f_3 f_4 (f''_3)^2 - f_1 f_2 f_3 f_4 f'_3 f''_4 + f_1 f_3 f_4 f_2^2 f'_4}{2 f_1 f_2 f_3 f_4^2}
\]

\[
+ \frac{-f_2 f_3 f_4 f_1 f_4' + f_2 f_3 f_4^2 f_1 f'_3}{2 f_1 f_2 f_3 f_4^2}.
\]

If \( f_1 \) and \( f_4 \) are given arbitrarily, this is the standard form showing the local solution exists. Then we note that we can choose \( \omega = dx^1 \), and \( a \) and \( b \) are given by

\[
a = \frac{f_1 f_2 f_3 (f'_4)^2 + f_2 f_3 f_4 f_4 f'_4 + f_1 f_2 f_4 f_3 f'_4 - f_1 f_3 f_4 f'_2 f'_4 - 2 f_1 f_2 f_3 f_4 f''_4}{4 f_1^2 f_2^2 f_3 f_4},
\]

\[
b = \frac{f_1 f_2 f_3 (f'_4)^2 + f_2 f_3 f_4 f_4 f'_4 + f_1 f_2 f_4^2 f'_3 f'_4 - 2 f_1 f_2 f_3 f_4 f''_4}{2 f_1^2 f_2 f_3 f_4^2}.
\]

• Second QE condition:
Here we have our PDE systems as \( \text{Ri} = 2 \nabla \nabla \lambda \). We choose that \( \lambda \) is also the only function of \( x^1 \). Then we obtain

\[
\nabla \lambda = \lambda' dx^1
\]

\[
\nabla \nabla \lambda = -\left( f_1 \lambda' \frac{\lambda'}{2 f_1} - \lambda'' \right) dx^1 dx^1 + \frac{f_2 \lambda'}{2 f_1} dx^2 dx^2 + \frac{f_3 \lambda'}{2 f_1} dx^3 dx^3 + \frac{f_4 \lambda'}{2 f_1} dx^4 dx^4.
\]
This gives our PDE system

\[
\begin{align*}
  f_2'' &= \frac{f_1 f_3 f_4 (f_2')^2 + f_2 f_3 f_4 f_1 f_2' - f_1 f_2 f_3 f_4 f_2' - f_1 f_2 f_3 f_4 f_2'}{2 f_1 f_2 f_3 f_4} - 4 f_1 f_2 f_3 f_4 f_2' \lambda' \\
  f_3'' &= \frac{f_1 f_2 f_4 (f_3')^2 + f_2 f_3 f_4 f_1 f_3' - f_1 f_3 f_4 f_2 f_3' - f_1 f_2 f_3 f_4 f_3'}{2 f_1 f_2 f_3 f_4} - 4 f_1 f_2 f_3 f_4 f_3' \lambda' \\
  f_4'' &= \frac{f_1 f_2 f_3 (f_4')^2 + f_2 f_3 f_4 f_1 f_4' - f_1 f_3 f_4 f_2 f_4' - f_1 f_2 f_3 f_4 f_4'}{2 f_1 f_2 f_3 f_4} - 4 f_1 f_2 f_3 f_4 f_4' \lambda' \\
  \lambda'' &= \frac{f_1 (f_2 f_3 f_4' + f_2 f_3 f_4' + f_2 f_3 f_4') + 2 \lambda' (f_1 f_2 f_3 f_4 + f_1 f_2 f_3 f_4 + f_1 f_2 f_3 f_4 + f_1 f_2 f_3 f_4)}{4 f_1 f_2 f_3 f_4}.
\end{align*}
\]

This seems complicated but, in fact, in terms of \( f_2 \) we can resolve it. So denoting \( f = f_2 \) and \( \nu = \sqrt{1 + n^2 + m^2} \) we obtain

\[
\begin{align*}
  f_1 &= c_1 c_2 c_3 f^{-\nu} (f')^2 \\
  f_3 &= c_2 f^m \\
  f_4 &= c_3 f^m \\
  \lambda &= \frac{1 + n + m + \nu}{4} \ln(f).
\end{align*}
\]
BIBLIOGRAPHY


A Basic notions of Riemannian geometry

A.1 CURVATURE OF RIEMANNIAN MANIFOLDS

We assume that \((M, g)\) will be a smooth unbounded \((n \geq 2)\)-dimensional Riemannian manifold. We denote its curvature tensor, Ricci tensor and scalar curvature, respectively, with \(R = R^h_{ijk}, Ri = R^n_{ijk}\) and \(sc = R^k_k\).

- The Riemannian curvature tensor \(R\) shows how far the metric tensor is not locally isometric to that of Euclidean space. So it is intuitively the amount that a geometric object, such as a surface, deviates from a flat plane, or a curve as in a line.

- The Ricci curvature \(Ri\) shows the difference in volume from a narrow conical piece of a standard ball in Euclidean space for the geodesic ball in a Riemannian manifold. This tensor provides a way of measuring the difference between Riemann’s geometry and the regular Euclidean \(n\)-dimensional geometry.

- The curvature scalar, \(sc\), represents the volume that deviates from the standard ball in the space of the Euclidean by a geodesic ball in a Riemannian manifold.

For general tensor \(A\) of type \((m, n)\), the Ricci identity has the form

\[ A^{i_1\cdots i_m}_{i_1\cdots i_m} - A^{i_1\cdots i_m}_{i_1\cdots i_m} = \sum_{q=1}^n A^{i_1\cdots i_q\cdots i_{q+1}\cdots i_m}_{i_1\cdots i_q\cdots i_{q+1}\cdots i_m} R^q_{i_1\cdots i_q} - \sum_{p=1}^m A^{i_1\cdots i_p\cdots i_{p+1}\cdots i_m}_{i_1\cdots i_p\cdots i_{p+1}\cdots i_m} R^p_{i_1\cdots i_p} . \]  

(A.1)

If the Ricci identity is used twice, the following consequences are identified.

\[ u^k_{i_1 i_2} = u^k_{j_1 j_2} + u^k_{j_1} R^h_{j_2 i_1} - u^h_{j_2} R^k_{i_1} - u^h_{j_1} R^k_{i_2} - u^k_{j_2} R^h_{i_1} ; \]

\[ u^k_{j_1 j_2} = u^k_{i_1 i_2} + u^k_{i_2} R^h_{i_1 j_2} - u^h_{i_1} R^k_{j_2} - u^h_{i_2} R^k_{j_1} - u^k_{i_2} R^h_{i_1} ; \]  

(A.2)

The Bianchi identity is

\[ R^i_{jk \ell} + R^i_{k j \ell} + R^i_{j k \ell} = 0 \]

\[ R^h_{ijkl} + R^h_{jilk} + R^h_{jkil} = 0 . \]  

(A.3)

By multiplying above equation by \(g^{hk}\), we have

\[ Ri_{ij\ell} - Ri_{i\ell j} = R^h_{ij} , \]

which then implies that

\[ sc_k = 2 (\text{div}(Ri))_k = 2 R^n_{k j} . \]  

(A.4)
A.1.1 On Two-dimensional manifold

In a two-dimensional manifold the curvature tensor can be written as follows:

\[
R_{ijk}^\ell = \kappa (g_{\ell k} g_{jk} - g_{\ell j} g_{ik}) \\
R_{ij}^k = \kappa (g_{jk} \delta_i^\ell - g_{ik} \delta_j^\ell) \\
\epsilon^\ell_{ij} R_{ijk}^\ell = -\kappa \epsilon_{ik} .
\]

(A.5)

Here \(\kappa\) is the Gaussian curvature.

A.1.2 On Three-dimensional manifold

In a three-dimensional manifold we have:

\[
R_{ijk}^\ell = R_{i\ell}^g j_k + R_{i\ell}^g k_j - R_{i\ell}^g j_k - R_{i\ell}^g k_j - \frac{sc}{2} (g_{\ell k} g_{jk} - g_{\ell j} g_{ik}) \\
R_{ij}^k = R_{i}^{\ell} g_{jk} - R_{j}^{\ell} g_{ik} + R_{j}^{\ell} \delta_i^k - R_{i}^{\ell} \delta_j^k - \frac{sc}{2} (g_{jk} \delta_i^\ell - g_{ik} \delta_j^\ell) .
\]

Moreover, by various (anti)symmetries

\[
\epsilon^{\ell h} R_{ijk}^\ell = -\epsilon^{\ell h} (R_{i\ell}^g j_k + R_{i\ell}^g k_j - R_{i\ell}^g j_k - R_{i\ell}^g k_j - \frac{sc}{2} (g_{\ell k} g_{jk} - g_{\ell j} g_{ik})) \\
\epsilon^{\ell m} g^{\ell h} R_{ijk}^\ell = -\epsilon^{\ell m} (R_{i\ell}^g j_k + R_{i\ell}^g k_j - R_{i\ell}^g j_k - R_{i\ell}^g k_j - \frac{sc}{2} (g_{jk} \delta_i^\ell - g_{ik} \delta_j^\ell)) \\
\epsilon^{\ell h} u^k (R_{ij}^{\ell} g_{jk} + R_{jk}^{\ell} g_{ij}) = 0 \\
\epsilon^{\ell h} u^k g_{\ell l} g_{jk} = 0 .
\]

(A.6)

Then, by applying (A.4), we obtain

\[
\epsilon^{\ell m} g^{\ell h} R_{ijk}^\ell = -\epsilon^{\ell m} R_{i\ell k} .
\]

(A.7)

A.2 USEFUL FORMULAS ON RIEMANNIAN MANIFOLDS

The \(T\) operation denoted to make a transpose, is closely related to the concept of symmetry. For \((0,2)\)-tensor \(u_{ij}\), the transpose is \(u_{ji}\). We regard \(\nabla u\) as a tensor of type \((0,2)\), so it can be presented as

\[
(\nabla u)_{ij} = u_{ij} .
\]

The transpose is now well defined as follows

\[
T_1^1(M) \xrightarrow{\sigma} T_0^0(M) \xrightarrow{T} T_0^0(M) \xrightarrow{\iota} T_1^1(M) ,
\]

that gives the formula

\[
(\nabla u)_{ji} = ((\nabla u)_{ij})^T = g^{jk} g_{ji} u_k^l .
\]

Lemma A.2.1

\[
\text{div}(u \otimes u) = u \nabla u + u \text{div}(u) \\
\text{div}(u \nabla u) = g(u, \text{grad}(\text{div}(u))) + g(\nabla u, \nabla u) + R(u, u) \\
\text{div}(\text{div}(Su)) = 2 \Delta(\text{div}(u)) + 2 \text{div}(R(u)) .
\]

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Proof. By definition, the first formula can be derived directly. In the third formula, we have
\[ \text{div}(u \nabla u) = u_j^i u_{ij}^l + u^i u^j_{ij}. \]
By applying (A.1), the above formulation can be rewritten as
\[
\text{div}(u \nabla u) = u_j^i u_{ij}^l + u^i u^j_{ij} + u^i u^h R_{ihj}
= g(\nabla u, \nabla u) + g(u, \text{grad(div}(u))) + R(u, u) ,
\]
where \( g(\nabla u, \nabla u) = \text{tr}((\nabla u)^2) \).
By definition \( S u = \nabla u + (\nabla u)^T \), so we prove
\[
\text{div(div}(((\nabla u)^T)) = \Delta(\text{div}(u)) + \text{div}(R(u)) . \tag{A.9}
\]
As in formula (A.1), we get
\[
(\text{div}(((\nabla u)^T)))^i = \text{div}(g^{hi} u_{j}^l) = g^{ih} u_{hj}^l = g^{ih} u^j_{ih} - g^{ih} u^j_{h} R_{i}^l
= (\text{grad(div}(u)))^i + (R(u))^i .
\]
Accordingly,
\[
\text{div(div}(((\nabla u)^T)) = \Delta(\text{div}(u)) + \text{div}(R(u)) .
\]
Next, we compute
\[
\text{div(div}(\nabla u)) = \text{div(div}(g^{hi} u_{j}^l)) = \text{div}(g^{hi} u_{j}^l) = g^{hi} u_{hji} .
\]
By using the Ricci identity for the \((1,1)\)-tensor, we get
\[
u_{hji} = (u_{h,j}^i)_i = u_{h,j}^i + u_{h,\ell}^i R_{j}^l_{h} - u_{h}^j R_{i}^l_{j}
= (u_{h,j}^i - u_{h,\ell}^i R_{i}^l_{h})_j + u_{h,\ell}^i R_{j}^l_{h} - u_{h}^j R_{i}^l_{j}
= u_{h,j}^i + u_{h,\ell}^i R_{i}^l_{h} + u_{h,\ell}^i R_{j}^l_{h} + u_{h}^j R_{i}^l .
\]
However,
\[
g^{hi} (u_{h,\ell}^i R_{j}^l_{h} + u_{h}^j R_{i}^l) = 0 ,
\]
so that
\[
\text{div(div}(\nabla u)) = g^{hi} u_{hji}^l + u_{h,\ell}^i R_{j}^l + u_{h}^j R_{i}^l .
\]
Moreover, we have
\[
\Delta(\text{div}(u)) = \text{div(\text{grad}(u_{j}^i))) = \text{div}(g^{hi} u_{ji}^l) = g^{hi} u_{hij}
\text{div}(R(u)) = \text{div}(u_{\ell}^i R_{j}^l) = u_{j}^i R_{i}^l + u_{\ell}^i R_{j}^l ,
\]
which yields the desired result. \(\square\)
First, the vector fields and functions are defined on the manifold. Then we must use rifsimp to obtain our differential equations. The DifferentialGeometry, LinearAlgebra, PDEtools, DEtools, Tensor and Tools packages can be used in MAPLE software. An example for RR is given here:

First load some packages.

\begin{verbatim}
with(LinearAlgebra); with(DifferentialGeometry); with(Tensor); with(Tools);
with(PDEtools); with(DEtools);
\end{verbatim}

Define the coordinates in $\mathbb{R}^3$.

\begin{verbatim}
DGsetup([x1,x2,x3],M3):
\end{verbatim}

Express metric components.

\begin{verbatim}
declare(f1(x1),f2(x1),f3(x1),h2(x2),h3(x2),q3(x3));
\end{verbatim}

Standard Riemannian vectors and covectors are defined in $\mathbb{R}^3$.

\begin{verbatim}
lis0:=GenerateTensors([lis,lis]); lis1:=GenerateTensors([lv,lis]);
\end{verbatim}

Metric matrix in $\mathbb{R}^3$.

\begin{verbatim}
ma:=Matrix([[f1(x1),0,0],[0,f2(x1),h2(x2)],[0,0,f3(x1).h3(x2).q3(x3)]]):
yk3:=IdentityMatrix(3):
ma0:=f1(x1).h1(x2).q1(x3).yk3:
ke:=convert(ma,list):
\end{verbatim}

Metric and christoffel symbols in $\mathbb{R}^3$.

\begin{verbatim}
g:=DGzip(ke,lis0,"plus");
ginv:=InverseMatrix(g):
chr:=Christoffel(g):
\end{verbatim}

Ricci curvature and Scalar curvature with their covariant derivative.

\begin{verbatim}
ri:=RicciTensor(g): ria:=RaiseLowerIndices(ginv,ri,[1]):
rib:=RaiseLowerIndices(ginv,ria,[2]):
dri:=CovariantDerivative(ri,chr): sc:=simplify(RicciScalar(g)):
dsc:=simplify(CovariantDerivative(sc,chr)):
\end{verbatim}

One form $\beta$ as formula (4.2).

\begin{verbatim}
\beta:=simplify(CovariantDerivative(ln(sc),chr)):
\end{verbatim}

PDE system (4.4).

\begin{verbatim}
t0:=simplify(evalDG(sc.dri-ri &t dsc)):
\end{verbatim}

The number of coefficients in the PDE system (4.4).

\begin{verbatim}
lt0:=DGinfo(t0,"CoefficientList","all"): numelems(lt0);
\end{verbatim}
The number of coefficients in the covariant derivative of scalar curvature.
\( l0 := DGinfo(dsc,"CoefficientList","all") \): numelems(ldsc);

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All coefficients in the covariant derivative of scalar curvature.
\( ap1 := op(2,simplify(ldsc[1])) \): \( ap2 := op(2,simplify(ldsc[2])) \):

All cases that lead to the most general solutions.
\( va := rifsimp([op(lt0),f1(x1)\neq 0,f2(x1)\neq 0,f3(x1)\neq 0,h2(x2)\neq 0,h3(x2)\neq 0,q3(x3)\neq 0,ap1],casesplit) \):
\( va0 := rifsimp([op(lt0),f1(x1)\neq 0,f2(x1)\neq 0,f3(x1)\neq 0,h2(x2)\neq 0,h3(x2)\neq 0,q3(x3)\neq 0,ap2],casesplit) \):
\( vas1 := copy(va[1]) \): \( vas2 := copy(va[2]) \):

Solve the above PDE system.
\( va0 := vas1[\text{Solved}] \): \( va0b := vas2[\text{Solved}] \):

\[
va0 := \begin{bmatrix}
h_{3_{x_2,x_2}} &= \frac{2 f_1 f_2 f_3 f_{3_{x_1}} h_2 h_3^2 + f_1 f_2 f_3 f_{3_{x_1}} h_3 h_{2_{x_2}} h_{3_{x_2}} - f_1 f_2 f_3 f_{3_{x_1}} h_2 h_3^2 - f_2 f_3 f_{3_{x_1}} h_2 h_3^2}{2 f_1 f_2 f_3 f_{3_{x_1}} h_2 h_3}, \\
f_{2_{x_1,x_1}} &= \frac{f_2 f_3 f_{1_{x_1}} f_{3_{x_1}} + 2 f_1 f_2 f_{3_{x_1}} - f_1 f_3 f_{2_{x_1}} f_{3_{x_1}}}{2 f_1 f_2 f_3}, \\
f_{3_{x_1,x_1}} &= \frac{f_3 f_{1_{x_1}} f_{3_{x_1}} + 2 f_1 f_3 f_{3_{x_1}}}{4 f_1 f_3 h_2 h_3} \end{bmatrix},
\[
va0b := \begin{bmatrix}
h_{3_{x_2,x_2}} &= -\frac{f_2 f_3 f_{3_{x_1}} h_2^2 h_3^2 - 3 f_1 f_3^2 h_2 h_3^2 - 2 f_1 f_3^2 h_3 h_{2_{x_2}} h_{3_{x_2}}}{4 f_1 f_3 h_2 h_3}, \\
f_{3_{x_1,x_1}} &= \frac{f_3 f_{3_{x_1}} (3 f_1 f_{3_{x_1}} + 2 f_1 f_{3_{x_1}})}{4 f_1 f_3}, f_{2_{x_1}} = \frac{f_2 f_{3_{x_1}}}{2 f_3} \end{bmatrix}
\]

check if solutions are solved by our PDE system.
\( simplify(dsubs(va0b,lhs(va0[1])-rhs(va0[1]))) \);
\( simplify(dsubs(va0b,lhs(va0[2])-rhs(va0[2]))) \);
\( simplify(dsubs(va0b,lhs(va0[3])-rhs(va0[3]))) \);
0
0
0

Differential equations of our PDE system.
\( yh0 := va0[2] \): \( yh1 := va0[3] \):

\[
yh0 := f_{2_{x_1,x_1}} = \frac{f_2 f_{3_{x_1}} f_{3_{x_1}} + 2 f_1 f_2 f_{3_{x_1}} - f_1 f_3 f_{2_{x_1}} f_{3_{x_1}}}{2 f_1 f_2 f_3}
\]

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Solve the above PDE system.

\[ va1 := \text{dsolve}([yh0, yh1]); \]

\[ va1 := [ f_1 = f_1, f_2 = C_3, f_3 = e^{C_1(f \sqrt{f_1^dx_1})} C_1, [f_2 = f_2, f_1 = \frac{-C_3f_2^{2C_1}}{f_2}, f_3 = f_2^{-C_1} C_2] \]

Simplify \( va1 \) solutions.

\[ va2 := \text{subs}([C_1 = m, C_2 = c_1, C_3 = c_2], va1[2]); va3 := [va2[2][1], va2[3][1]]; \]

\[ va3 := [f_1 = \frac{c_2f_2^{2C_1}}{f_2}, f_3 = f_2^m c_1] \]

To have another metric component, \( h_3 \), Substitute \( va3 \) to \( va0[1] \) PDE system.

\[ yh2 := \text{dsubs}(va3, va0[1]); \]

\[ va4 := \text{subs}(C_1 = c_3, \text{dsolve}(yh2, h_2(x_2))); \]

\[ va5 := \text{simplify}(\text{dsubs}(h_3(x_2) = h(x_2)^m, va4), \text{power}, \text{symbolic}); \]

\[ va5 := h_2 = \frac{-c_2m^2 h_3^2}{h(m^2h - c_3c_2)} \]

Total solutions of PDE system (4.4).

\[ va6 := \text{op}(va3), va5, h_3(x_2) = h(x_2)^m]; \]

\[ va6 := [f_1 = \frac{c_2f_2^{2C_1}}{f_2}, f_3 = f_2^m c_1, h_2 = \frac{-c_2m^2 h_3^2}{h(m^2h - c_3c_2)}, h_3 = h^m] \]
Navier–Stokes equations on Riemannian manifolds
Maryam Samavaki *, Jukka Tuomela
University of Eastern Finland, Department of Physics and Mathematics, P.O. Box 111, FI-80101 Joensuu, Finland

Abstract
We study properties of the solutions to Navier–Stokes system on compact Riemannian manifolds. The motivation for such a formulation comes from atmospheric models as well as some thin film flows on curved surfaces. There are different choices of the diffusion operator which have been used in previous studies, and we make a few comments why the choice adopted below seems to us the correct one. This choice leads to the conclusion that Killing vector fields are essential in analyzing the qualitative properties of the flow. We give several results illustrating this and analyze also the linearized version of Navier–Stokes system which is interesting in numerical applications. Finally we consider the 2 dimensional case which has specific characteristics, and treat also the Coriolis effect which is essential in atmospheric flows.

1. Introduction
Navier–Stokes equations have been widely studied both from theoretical and applied points of view [15]. Perhaps the first paper where Navier–Stokes system was considered on Riemannian manifolds was [5]. In recent years there has been a growing interest of this problem, see [2,3,9,11,12,14] and the many references therein. There seems to be two different reasons for this interest. First are the atmospheric models where the curvature of earth matters if one wants to simulate the flow in very large domains or even on the whole earth. The second are the flows of very thin films on the curved surfaces. Although these two applications are physically very different they both lead naturally to the idea of formulating the Navier–Stokes equations on arbitrary Riemannian manifolds.

There have been different choices for the diffusion operator for the system on the manifolds, and we discuss first some reasons why we think that the choice adopted below is the appropriate one. The same choice is advocated also in [3,14]. It turns out that this choice of diffusion operator has important consequences on the qualitative and asymptotic properties of the flow, and our choice implies that Killing vector fields are essential in the analysis.

Our main results concern the decomposition of the flows to Killing component and its orthogonal complement. The Killing vector field is actually a solution to the Navier–Stokes system, but due to nonlinearity the orthogonal complement satisfies a different system. However, it is possible to derive similar a priori estimates for this complement than to the total flow. Interestingly similar conclusion remains valid when one replaces the diffusion operator with another operator and Killing fields with harmonic vector fields. Hence depending on the choice of the diffusion operator the solutions obtained are completely different asymptotically.

We will also analyze the linearized version of Navier–Stokes system. This is interesting at least from the point of view of numerical solution of Navier–Stokes system. Often one uses the idea of operator splitting in order to treat the linear
diffusion term and the nonlinear convection term differently (a thorough overview of numerical methods for Navier-
Stokes system is given in [6]). Then in some numerical methods one linearizes the convection term to advance the solution.
We show that also the linearized version respects this decomposition to Killing fields and the orthogonal complement.
Moreover it turns out that one can produce new solutions with Lie bracket. Given a solution to a linearized system
and a Killing field their bracket is also a solution to the linearized system. This is rather a technical result where we show
that various differential operators behave well with respect to bracket operation when one of the fields is a Killing field.
Since Killing fields seem to play perhaps even a surprisingly big role in this context one may wonder why they do not appear to be so important for Navier–Stokes equations in $\mathbb{R}^n$. Probably the main reason is that most of the natural
problems in $\mathbb{R}^n$ are boundary value problems and since the Killing fields typically do not satisfy the boundary conditions
they do not appear as solution candidates.

Since 2 dimensional manifolds are especially important in applications we analyze this special case more closely. In
particular the sphere is relevant in meteorological applications so we consider this in detail. It turns out that one can
decompose also the vorticity in the same way as the flow field itself. Since the vorticities of the Killing fields are simply
the first spherical harmonics one can use this to get good a priori estimates. Finally we consider the case of Navier–Stokes
on the sphere with the Coriolis term. In this case the Killing field along the latitudes is still a solution and asymptotically
the solutions approach it. Note that here again the asymptotic properties depend essentially on the choice of the diffusion
operator.

2. Model and the diffusion operator

The standard way to write the Navier–Stokes equations in $\mathbb{R}^n$ is as follows

$$u_t + u \nabla u - \mu \Delta u + \nabla p = f$$

$$\nabla \cdot u = 0$$

Let us formulate this on an arbitrary Riemannian manifold $M$ with Riemannian metric $g$. Let $\nabla$ now denote the covariant
derivative, and to avoid confusion we write $(\nabla f)^i = g^{ij} f_j$ for the gradient and $\text{div}(u) = tr(\nabla u) = u^i_i$ for the
divergence.\(^1\) The nonlinear term is now $(\Delta u)^i = g^{ij} u^k_{,jk}$ for the Laplacian of the scalar function is
This is perhaps mathematically natural, since this is in a sense the first thing that comes to mind, considering that the
problems in $\mathbb{R}^n$ are boundary value problems and since the Killing fields typically do not satisfy the boundary conditions
they do not appear as solution candidates.

Then for one form

$$(\Delta \alpha)^i_j = g^{ik} u^j_{,kj} - g^{ik} u^j_{,ik}$$

Then we can write

$$(\Delta \alpha)^i_j = g^{ik} u^j_{,kj} - g^{ik} u^j_{,ik} = g^{ik} u^j_{,kj} - \Delta (u)^i_j$$

Now using the Ricci identity (A.3) we obtain

$$\Delta (u) = \text{div}(\text{grad}(u)) = \text{div}(\text{grad}(u)) = \alpha(u)$$

where $\alpha$ is the Ricci tensor. In two and three dimensional cases we also have the familiar formulas

$$(\Delta (u) = \text{grad}(\text{div}(u)) - \text{Rot}(\text{rot}(u)))$$

$$(\Delta (u) = \text{grad}(\text{div}(u)) - \text{curl}(\text{curl}(u)))$$

where the operators rot, Rot and curl are defined in Appendix B.

However, one can argue that Hodge Laplacian is not appropriate for the present purposes either. Recall that a
(Newtonian) fluid is characterized by the fact that the stress tensor is a function of deformation rate tensor [16]. Classically
the deformation rate tensor is $1/2 (\nabla u + (\nabla u)^T)$. In the Riemannian case we set (omitting the factor $1/2$)

$$(\text{Sur})^i = g^{ik} u^j_{,kj} + g^{ik} u^j_{,ik}$$

Hence, as in [3, 14], we get the diffusion operator $Lu = \text{div}(Su)$.

\(^1\) Einstein summation convention is used where needed.
Lemma 2.1.

\[ Lu = \Delta u + \text{grad(div}(u)) = \text{R}(u) \]

Proof. First we compute

\[ (Lu)^2 = g^{jl}u_j^l u_k^l + g^{jl}u_j^k = (\Delta u)^2 + g^{jl}u_j^k \]

Using the Ricci Identity (A.3) we get

\[ g^{jl}u_j^k = g^{jl}u_j^l + g^{jl}(\text{grad(div}(u))) + (\text{R}(u)) \]

So the operators \( L \) and \( \Delta u \) give different signs for the curvature term. Summarizing we may say that Bochner Laplacian uses information about the whole of \( \text{div}(u) \) and ignores the curvature term, while \( L \) uses the symmetric part, and Hodge Laplacian uses the antisymmetric part. The sign of the curvature term is different in the symmetric and antisymmetric cases.

It seems to us that the operator \( L \) is physically most natural candidate for the diffusion operator because it most naturally generalizes the constitutive laws which are used in the Euclidean spaces. Also in [3] the authors come to the conclusion that \( L \) is the best choice. However, in the pioneering paper [5] the Hodge Laplacian is used. Also more recently in [2,9] \( \Delta u \) is used, and in [9] it is argued that \( \Delta u \) is actually an appropriate choice, at least in some situations. Finally in [11] Bochner Laplacian is used. We do not know how the choice of Bochner Laplacian or the Hodge Laplacian should be interpreted from the point of view of continuum mechanics.

Mathematically the choice of \( \Delta u \) is convenient because then one can use the de Rham complex and the resulting (Helmholtz) decompositions of various fields. When one uses \( L \) perhaps the (formally exact) compatibility complex for the operator \( S \) would be of interest. When the curvature is constant this complex is known as Calabi complex, [8]. We do not know if this complex has been used to study Navier–Stokes equations or if the appropriate complex has even been constructed in the general case.

So we take \( L \) as our diffusion operator and proceed our analysis with it:

\[ u_t + \nabla_v u - \mu Lu + \text{grad}(p) = 0 \]

\[ \text{div}(u) = 0 \] (2)

However, some of the results are valid whatever the choice of the Laplacian and we will analyze what kind of effect this choice has.

The actual existence and uniqueness of solutions to system (2) is a difficult problem even in classical context. For small times one can prove the existence and uniqueness of weak solutions under reasonable hypothesis. However, the uniqueness can fail, if the solutions are “too weak”, see the recent work of Buckmaster and Vicol [1]. To get global solutions one must assume rather restrictive hypothesis, and as is well-known the existence and precise nature of global solutions is the best choice. However, in the pioneering paper [5] the Hodge Laplacian is used. Also more recently in [2,9] \( \Delta u \) is used, and in [9] it is argued that \( \Delta u \) is actually an appropriate choice, at least in some situations. Finally in [11] Bochner Laplacian is used. We do not know how the choice of Bochner Laplacian or the Hodge Laplacian should be interpreted from the point of view of continuum mechanics.

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3. Preliminaries and notation

Let us now introduce some appropriate functional spaces, see [7] for more details. Let us define the \( L^2 \) inner product for functions and vector fields by the formulas

\[ (f, h) = \int_M f h \omega_M \]

\[ (u, v) = \int_M g(u, v) \omega_M \]

where \( \omega_M \) is the volume form (or Riemannian density if \( M \) is not orientable). This gives the norm \( \|u\|_2 = \sqrt{(u, u)} \).

Similarly we can introduce inner products for tensor fields. However, since we need this just for one forms and tensors of type (1, 1) we give the formulas only for this case. For one forms \( \alpha \) and \( \beta \) we can simply write \( g(\alpha, \beta) = g(\alpha \circ, \beta \circ) = g^i \alpha_i \beta_i \) Then let \( T \) be a tensor of type (1, 1); pointwise \( T \) can be interpreted as a map \( T : T_p M \to T_p M \). Let \( T^* \) be the adjoint, i.e.

\[ g(Tu, v) = g(u, T^*v) \]
we have the following classical characterization for harmonic vector fields
for all $v$ vector fields are much more common. Let us recall the following facts [10].

Definition 3.1. Vector field $u$ is parallel if $\nabla u = 0$, it is Killing if $Su = 0$ and it is harmonic, if $A_u u = 0$.
Equivalently we can say that $u$ is Killing, if
$$ g(\nabla_v u, w) + g(\nabla_w u, v) = 0 $$
for all $v$ and $w$. Note that $\text{div}(u) = 0$ for Killing fields because $\text{div}(u) = \frac{1}{2} \text{tr}(S_u)$. If $M$ is compact and without boundary
we have the following classical characterization for harmonic vector fields
$$ A_u u = 0 \iff \frac{\partial u}{\partial v} = 0 \iff \frac{\partial u}{\partial w} = 0 \iff \text{div}(u) = 0 $$
where $A_u$ is given in (1). Hence in particular
$$ g(\nabla_v u, w) - g(\nabla_w u, v) = 0 $$
for all $v$ and $w$ if $u$ is harmonic.

There are severe topological restrictions for the existence of parallel vector fields [17]. Killing vector fields and harmonic vector fields are much more common. Let us recall the following facts [10].

(i) If $M$ is $n$ dimensional then the Killing vector fields are a Lie algebra whose dimension is $\leq \frac{1}{2} n(n + 1)$ and the equality is attained for the standard sphere.

(ii) the space of harmonic vector fields is isomorphic to the first de Rham cohomology group of $M$. In particular this space is also always finite dimensional.

Lemma 3.2. Let $u$, $v$ and $w$ be vector fields and $\text{div}(v) = 0$. Then
$$ \int_M (g(\nabla_v u, w) + g(\nabla_w u, v))_{\omega_M} = 0 $$
In particular
$$ \int_M g(\nabla_u u, v)_{\omega_M} = 0 $$

Proof. Since
$$ \text{div}(g(u, v) v) = g(u, w)\text{div}(v) + g(\nabla_u u, w) + g(\nabla_v u, w) $$
the result follows from divergence theorem. □

Lemma 3.3. If $w$ is Killing and $\text{div}(u) = \text{div}(v) = 0$ then
$$ \int_M (g(\nabla_v u, w) + g(\nabla_w u, v))_{\omega_M} = 0 $$
and if $w$ is harmonic and $\text{div}(u) = \text{div}(v) = 0$ then
$$ \int_M (g(\nabla_v u, w) - g(\nabla_w u, v))_{\omega_M} = 0 $$
Proof. Using Lemma 3.2 and formula (3) we obtain
\[ \int_M (g(\nabla_v u, w) + g(\nabla_u v, w)) \omega_M = -\int_M (g(\nabla_v u, w) + g(\nabla_u v, w)) \omega_M = 0 \]
The proof of the second statement is analogous. □

From this we immediately get

Lemma 3.4. If either (i) \( u \) is Killing and \( \text{div}(v) = 0 \) or (ii) \( v \) is Killing and \( \text{div}(u) = 0 \) then
\[ \int_M g(\nabla_v u, u) \omega_M = \int_M g(\nabla_u v, u) \omega_M = 0 \]
and if \( u \) is harmonic and \( \text{div}(v) = 0 \) then
\[ \int_M g(\nabla_v u, u) \omega_M = 0 \]

Proof. Follows directly from previous Lemmas. □

Lemma 3.5. Let \( u \) and \( v \) be vector fields. Then
\[ \int_M g(\Delta u, v) \omega_M + \int_M g(\nabla_u \nabla v) \omega_M = 0 \]
\[ \int_M g(Lu, v) \omega_M + \frac{1}{2} \int_M g(S_u, S_v) \omega_M = 0 \]
Hence \( (\Delta u, v) = (u, \Delta v) \) and \( (Lu, v) = (u, Lv) \).

Proof. We compute
\[ \text{div}(u^k u^l g_{ij} v^j v^l) = g^{k^l} u^l g_{ij} v^j + g^{k^l} u^{l^i} v^j v^l = g(\Delta u, v) + g(\nabla_u, v) \]
\[ \text{div}(S_u v) = g^{k^l} u^l g_{ij} v^j + g^{k^l} u^{l^i} v^j + u^k v^j + u^l v^l = \frac{1}{2} g(S_u, v) + g(Lu, v) \]
The result now follows from the divergence theorem. □

Note that the above Lemma, the relationships between the Laplacians and the operator \( L \) imply that for divergence free vector fields
\[ |\int_M R(u, u) \omega_M| \leq \int_M g(\nabla_u, \nabla u) \omega_M \]
and
\[ \int_M R(u, u) \omega_M = 0 \quad \text{if } u \text{ is parallel} \]
\[ \int_M R(u, u) \omega_M = \int_M g(\nabla_u, \nabla u) \omega_M \quad \text{if } u \text{ is Killing} \]
\[ \int_M R(u, u) \omega_M = -\int_M g(\nabla_u, \nabla u) \omega_M \quad \text{if } u \text{ is harmonic} \]
So Killing vector fields and harmonic vector fields are at the “opposite extremes” with respect to curvature.

4. Solutions to Navier–Stokes system and Killing fields

Let us then start to analyze the properties of the solutions to (2). Let us first recall the following facts which are easy to check:

(i) if \( u \) is parallel and \( p \) is constant then \( (u, p) \) is a solution of Navier–Stokes equations with Bochner Laplacian.

(ii) if \( u \) is Killing and \( p = \frac{1}{2} g(u, u) \) then \( (u, p) \) is a solution of (2).

(iii) if \( u \) is harmonic and \( p = -\frac{1}{2} g(u, u) \) then \( (u, p) \) is a solution of Navier–Stokes equations with Hodge Laplacian.

Our first main result says that the component of any solution in the space of Killing fields remains constant.
Theorem 4.1. Let $u$ be a solution of (2) and $v$ be Killing. Then
\[
\frac{d}{dt}(u, v) = 0
\]

Proof. First
\[
\frac{d}{dt}(u, v) = (u_t, v) = -\left(\nabla_u u, v\right) + \mu \left(\Delta u, v\right) - \left(\text{grad}(p), v\right)
\]
Then $\left(\nabla_u u, v\right) = 0$ by Lemma 3.4, $\left(\mu u, v\right) = 0$ by Lemma 3.5 and because $v$ is Killing, and $\left(\text{grad}(p), v\right) = -\left(p, \text{div}(v)\right) = 0$ because $\text{div}(v) = 0$.

In other words any solution can be decomposed as $u = u^\perp + u^+$ where $u^\perp$ is Killing and $u^+$ is orthogonal to Killing fields. One may view $u^+$ as a projection of the initial condition to the space of Killing fields. But then precisely with the same argument we get

Theorem 4.2. Let $u$ be a solution of Navier–Stokes equations with Hodge Laplacian and let $v$ be harmonic. Then
\[
\frac{d}{dt}(u, v) = 0
\]

Proof. Now we have
\[
\frac{d}{dt}(u, v) = (u_t, v) = -\left(\nabla_u u, v\right) + \mu \left(\Delta u, v\right) - \left(\text{grad}(p), v\right)
\]
Evidently $\left(\Delta u, v\right) = (u, \Delta u) = 0$ and $\left(\nabla_u u, v\right) = 0$ by Lemma 3.4.

Let us then continue with system (2). The whole dynamics of the solution $u = u^\perp + u^+$ thus happens in the component $u^+$. Then writing $p = p_\perp + p_+$ where $p_\perp = \frac{1}{2} g(u^\perp, u^\perp)$ we get the following system for $u^+$:
\[
\begin{align*}
 u^+_{t} & + \nabla_{u^+} u^+ + \nabla_{u^+} u^+ + \nabla_{u^+} u^+ - \mu (u^+ + \text{grad}(p_+)) = 0 \\
 \text{div}(u^+) &= 0
\end{align*}
\]

(4)

In the absence of forces acting on the system one expects that $u^+$ would approach zero when $t \to \infty$. To state this precisely we need a short digression. Let us first define
\[
\begin{align*}
 V_p &= \{u \in H^1(M) \mid \|u\| = 1, \ (u, v) = 0 \text{ for all parallel } v\} \\
 V_k &= \{u \in H^1(M) \mid \|u\| = 1, \ (u, v) = 0 \text{ for all Killing } v\} \\
 V_h &= \{u \in H^1(M) \mid \|u\| = 1, \ (u, v) = 0 \text{ for all harmonic } v\}
\end{align*}
\]

Then we can set
\[
\begin{align*}
 \alpha_p &= \inf_{u \in V_p} \int_M g(\nabla u, \nabla u) \omega_M \\
 \alpha_k &= \inf_{u \in V_k} \int_M g(S_u, S_u) \omega_M \\
 \alpha_h &= \inf_{u \in V_h} \int_M \left(g(A_u, A_u) + \text{div}(u)^2\right) \omega_M
\end{align*}
\]

What are the values of these constants? We could not find anything in the literature. The book Hebey [7] treats extensively topics which are directly related, but everything is about functions, not vector fields. Anyway let us show that these numbers actually are positive.

Theorem 4.3. The numbers $\alpha_p, \alpha_k$ and $\alpha_h$ are strictly positive and we have Poincaré type inequalities:
\[
\begin{align*}
 \alpha_p \int_M g(u, u) \omega_M &\leq \int_M g(\nabla u, \nabla u) \omega_M , \quad \forall u \in V_p \\
 \alpha_k \int_M g(u, u) \omega_M &\leq \int_M g(S_u, S_u) \omega_M , \quad \forall u \in V_k \\
 \alpha_h \int_M g(u, u) \omega_M &\leq \int_M \left(g(A_u, A_u) + \text{div}(u)^2\right) \omega_M , \quad \forall u \in V_h
\end{align*}
\]
Then we compute

\[ \lim_{k \to \infty} \int_M g(\nabla u_k, \nabla u_k)_{\text{vol}} = \inf_{u \in V} \int_M g(\nabla u, \nabla u)_{\text{vol}} \]

By Rellich–Kondrakov Theorem there is a subsequence (still denoted by \( u_k \)) such that \( u_k \) converges weakly in \( H^1(M) \) and strongly in \( L^2(M) \). Strong convergence implies that the limit \( \hat{u} \in V_p \) and the weak convergence that

\[ \int_M g(\nabla \hat{u}, \nabla \hat{u})_{\text{vol}} \leq \lim_{k \to \infty} \int_M g(\nabla u_k, \nabla u_k)_{\text{vol}} \]

Since \( \int_M g(\nabla \hat{u}, \nabla \hat{u})_{\text{vol}} > 0 \), \( \alpha_p > 0 \). □

**Theorem 4.4.** Let \( u^t \) be a solution of (4). Then

\[ \|u^t\|^2 \leq C e^{-\mu_a t} \]

**Proof.** Lemma 3.2 and formula (3) imply that

\[ \int_M g(\nabla u^t, u^t)_{\text{vol}} = \int_M g(\nabla u^t, u^t)_{\text{vol}} = \int_M g(\nabla u^t, u^t)_{\text{vol}} = 0 \]

Then combining Theorem 4.3 and Lemma 3.5 we have

\[ \frac{d}{dt} \|u^t\|^2 = -\mu \int_M g(S_{u}, S_{u})_{\text{vol}} \leq -\mu \alpha_k \int_M g(u^t, u^t)_{\text{vol}} = -\mu \alpha_k \|u^t\|^2 \]

As a consequence we obtain

\[ \nabla u - \mu Lu + \text{grad}(\rho) = 0 \]

are precisely the Killing vector fields.

As a consequence we see that the asymptotic behavior of solutions is totally different for \( L \) and \( \Delta M \). For example there are no harmonic vector fields on the sphere so that in the absence of forces all solutions tend to zero if Hodge Laplacian is used. But with the system (2) the solutions tend to some Killing field. But the Killing fields on the sphere correspond to the rotating motion which is physically very natural. We think that this is one more argument in favor of \( L \) compared to \( \Delta M \), in addition to the discussion in [3].

At least for numerical purposes it is essential to analyze the elliptic equation satisfied by the pressure.

**Lemma 4.5.**

\[ -\Delta p - \text{tr}((\nabla u)^2) - R(\nabla u, u) + 2 \mu \text{div}(\nabla u) = 0 \]

**Proof.** First \( \text{div}(\Delta u) = g^{ij} u_{ij} \); then applying the formula (A.4) we get

\[ u_{ij} = u_{ij} + u_{ji} R_{ij} + u_{ij} R_{ij} - u_{ij} R_{ij} \]

But \( g^{ij}(u_{ij} R_{ij} - u_{ij} R_{ij}) = 0 \) which implies that

\[ \text{div}(\Delta u) = g^{ij} u_{ij} + g^{ij} R_{ij} + u_{ij} R_{ij} + u_{ij} R_{ij} + u_{ij} R_{ij} \]

\[ = \Delta(\text{div}(u)) + \text{div}(\nabla R(u)) \]

Consequently

\[ \text{div}(Lu) = 2\Delta(\text{div}(u)) + 2 \text{div}(\nabla R(u)) \] (5)

Then we compute

\[ \text{div}(\nabla u) = u_{ij} u_{ij} + u_{ij} u_{ij} = u_{ij} u_{ij} + R_{ij} u_{ij} + u_{ij} \]

\[ = g(\text{grad}(\text{div}(u)), u) + R(\nabla u, u) + \text{tr}(\nabla u^2) \]

where we have used the formula (A.3). Then using the fact that \( \text{div}(u) = 0 \) gives the result. □

Note that the formula (5) implies that \( \text{div}(R(u)) = 0 \) for Killing vector fields. If Hodge Laplacian is used the same computations give

\[ -\Delta p - \text{tr}((\nabla u)^2) - R(\nabla u, u) = 0 \]
Hence there is no term which depends on the diffusion. So for manifolds where the term \( \text{div}(R(u)) \) is big, for example manifolds whose curvature changes fast, the pressure given by \( L \) and \( \Delta_H \) should be considerably different.

5. Linearized Navier–Stokes

In the numerical solution of Navier–Stokes equations one has to deal with the linearized system so let us consider this case also.

\[
\begin{align*}
    u_t + \nabla_u v + \nabla_v u &= -\mu Lu + \text{grad}(p) \\
    \text{div}(u) &= 0
\end{align*}
\]  

Typically one can think of \( v \) as the initial condition, and then one solves the same linear system for a few time steps. Interestingly the solution to the linearized system also preserves its aspect with respect to the space of Killing vector fields.

Theorem 5.1. Let \( u \) be a solution of (6) and \( w \) be Killing. Then

\[
\frac{d}{dt} (u, w) = 0
\]

Proof. As in Theorem 4.1 we first obtain

\[
\frac{d}{dt} (u, w) = -\langle \nabla_u v, w \rangle - \langle \nabla_v u, w \rangle + \mu \langle Lu, w \rangle - \langle \text{grad}(p), w \rangle
\]

Again \( (Lu, w) = (u, LW) = 0 \) and \( \langle \text{grad}(p), w \rangle = -\langle p, \text{div}(w) \rangle = 0 \) because \( w \) is Killing. But then

\[
\langle \nabla_u v, w \rangle + \langle \nabla_v u, w \rangle = 0
\]

using Lemma 3.3. □

This result is important because quite often in practice one uses either \( \nabla_u \) or \( \nabla_v \) for the linearized convection term because this is easier to implement. However, in that case the inner product is not preserved so that the computed solution is not qualitatively correct. This strongly suggests that a better solution is obtained when the full linearized convection term is used.

Let us then write as before \( u = u^K + u^\perp \) and \( v = v^K + v^\perp \). The pressure term can now be decomposed as \( p = p_\perp + p_\parallel = g(u, v) + p_\parallel \). Let us further denote \( f = -\nabla_u v^K - \nabla_v u^K \). Then we can write the linear system for \( u^K \) as follows:

\[
\begin{align*}
    u^K_t + \nabla_u u^K + \nabla_v u^K &= -\mu u^K + \text{grad}(p^K) \\
    \text{div}(u^K) &= 0
\end{align*}
\]

In this case the norm of the solution does not necessarily diminish because now \( f \) acts like a forcing term.

Theorem 5.2. Let \( u^K \) be a solution of (7). Then

\[
\frac{d}{dt} \|u^K\|^2 \leq -\mu c_\perp \|u^K\|^2 + 2f, u^K \rangle - 2\langle \nabla_{u^K} v^K, u^K \rangle
\]

Proof. First we have

\[
\frac{1}{2} \frac{d}{dt} \langle u^K, u^K \rangle = \langle f, u^K \rangle - \langle \nabla_{u^K} u^K, u^K \rangle - \langle \nabla_{u^K} v^K, u^K \rangle
\]

\[
- \langle \nabla_{u^K} v^K, u^K \rangle - \langle \nabla_{u^K} u^K, u^K \rangle + \mu \langle Lu^K, u^K \rangle - \langle \text{grad}(p), u^K \rangle
\]

First \( \langle \text{grad}(p), u^K \rangle = -\langle p, \text{div}(u^K) \rangle = 0 \) as usual. Now \( g(\nabla_{u^K} v^K, u^K) = 0 \) because \( v^K \) is Killing and \( \langle \nabla_{u^K} u^K, u^K \rangle = \langle \nabla_{u^K} u^K, u^K \rangle = 0 \) by Lemma 3.2. Hence

\[
\frac{1}{2} \frac{d}{dt} \langle u^K, u^K \rangle = \langle f, u^K \rangle - \langle \nabla_{u^K} v^K, u^K \rangle + \mu \langle Lu^K, u^K \rangle
\]

Then combining Theorem 4.3 and Lemma 3.5 we get the result. □

Hence in this case the norm might grow for some choices of \( v \). Note, however, that in the intended application the norm actually diminishes for small time. Typically \( v \) is considered as the initial condition for \( u \) so that \( u(0) = u^K + u^\perp(0) = v^K + v^\perp \). Hence if we write

\[
\beta(t) = \langle f, u^K \rangle - \langle \nabla_{u^K} v^K, u^K \rangle
\]

then clearly \( \beta(0) = 0 \). But this stability for small time is sufficient in practice because one only takes a few time steps with same \( v \).
6. New solutions from old

One interesting property of the linearized problem is that one can produce new solutions with the bracket. In the proof of the result we need the following property of Killing vector fields [10]. If \( \nu \) is a Killing vector field then

\[
\nabla_X R = R(Y, v)X
\]

which can be written with indices as

\[
u_{i|} = -\nu^i R_{i|}
\]

(6)

**Theorem 6.1.** Let \((u, p)\) be a solution of (6) where we suppose that \( \nu \) is Killing. Then

\[
(\hat{u}, \hat{p}) = \left( [u, \nu], -g(\text{grad}(p), \nu) \right)
\]

is also a solution of (6).

**Proof.** First it is straightforward to check that \( \text{div}(\hat{u}) = \text{div}([u, \nu]) = 0 \). For \( \hat{u} \) we have the following equation:

\[
u + [\nabla \nu, u] + [V \nu, v] - [\Delta \nu, \nu] - [R \nu, v] + [\text{grad}(p), v] = 0
\]

We have to verify the following claims.

**Claim 1.** \([\text{grad}(p), \nu] = \text{grad}(\hat{p})\).

\[
[[\text{grad}(p), \nu]] = (\nabla_{\text{grad}(\hat{p})}) - (\nabla_{\nu}(\text{grad}(p))) = g^i p v_i - g^k p v_k = -g^i p_i v^k - g^i p_k v^i - g^i \tilde{v}^k i - g^k \tilde{v}^i i
\]

(8)

**Claim 2.** \(\nabla_X [u, \nu] = [\nabla_X u, \nu] + [\nu, \nabla_X u] - 2\nabla_{[\nu, u]} v\). We have

\[
(\nabla_X [u, \nu])^j = (\nabla_X u)^j +\nu^j (\nabla_X v)^i
\]

**Claim 3.** \(\nabla_{[\nu, v]} u = [\nabla_{\nu} u, v] + [\nabla_{\nu} v, u] = [\text{grad}(u), v] + [\text{grad}(v), u] = [\nabla_{\nu} u, v] + [\nabla_{\nu} v, u]\)

**Claim 4.** \(\Delta_\nu u = \Delta_\nu^u v\).

In coordinates we have

\[
[[\Delta_\nu u, \nu]]^i = \gamma_{ijk} u^j \nu^k - \nu^i u_{ijk}
\]

(9)

Let \( \Delta_\nu u + R(\nu) \) will show that \( \Delta = 0 \). Using the formula (9.4) we obtain

\[
T^i = \gamma_{ijk} u^j \nu^k - v^i u_{ijk} - 2v^i u_{ijk} - 2v^i u_{ijk} + v^i R_{ijk}^l + v^i R_{ijk}^l
\]

Since \( \Delta = 0 \) and \( R(\nu) = 0 \) we have

\[
\gamma_{ijk} v^j u_{ijk} + \gamma_{ljk} v^l u_{ijk} = \gamma_{ijk} v^j u_{ijk} + \nu^i u_{ijk} = (\nabla_{\nu} u)^i + (\nabla_{\nu} v)^j = 0
\]

Using this and formula (8) thus yields

\[
T^i = -\gamma_{ijk} u^j \nu^k + v^i u_{ijk}
\]

Then we can write

\[
2v^i u_{ijk} = v^i u_{ijk} + v^i u_{ijk} - v^i u_{ijk}
\]
so finally using the Killing property
\[ T^i = -g^{bh}(v_{\text{A}}u^i_{\text{A}} + v_{\text{B}}u^i_{\text{B}}) = g^{bh}v_{\text{A}}u^i_{\text{A}} - g^{bh}v_{\text{B}}u^i_{\text{B}} = 0 \]

Claim 5. If \( u \) is the solution of \( \text{Ri} = [\text{Ri}(u), v] \),

First we compute
\[
(\text{Ri}(u), v)^{\prime} = \text{Ri}_{\alpha}u^\alpha v^i - \text{Ri}_{\beta}u^\beta v^i - \text{Ri}_{\gamma}u^\gamma v^i
\]
\[
= \text{Ri}(u, v)^{\prime} + \text{Ri}_{\gamma}u^\gamma v^i - \text{Ri}_{\beta}u^\beta v^i - \text{Ri}_{\gamma}u^\gamma v^i
\]

Hence we need to prove that
\[ T_b^i = \text{Ri}_{\alpha}u^\alpha v^i - \text{Ri}_{\beta}u^\beta v^i - \text{Ri}_{\gamma}u^\gamma v^i = 0 \] (9)

Now formula (8) implies that
\[ v_i^{\prime\prime} = v_i^\alpha \text{Ri}_{\alpha} + v^i \text{Ri}_{\text{B}}u^\text{B} \]

Applying this to the formula (9) we get
\[ T_b^i = \text{Ri}_{\alpha}u^\alpha v^i - \text{Ri}_{\beta}u^\beta v^i - \text{Ri}_{\gamma}u^\gamma v^i \]

Using Ricci identity (A.3) and the fact that \( v \) is Killing we get
\[
g^\mu v_i^{\prime\prime} = g^\mu v_i^{\prime\prime} + \text{Ri}_{\alpha}u^\alpha v^i + \text{Ri}_{\beta}u^\beta v^i + \text{Ri}_{\gamma}u^\gamma v^i
\]
\[
= -g^\mu v_i^{\prime\prime} + \text{Ri}_{\gamma}u^\gamma v^i + \text{Ri}_{\beta}u^\beta v^i + \text{Ri}_{\gamma}u^\gamma v^i
\]
\[
= -g^\mu v_i^{\prime\prime} + \text{Ri}_{\gamma}u^\gamma v^i + \text{Ri}_{\beta}u^\beta v^i + \text{Ri}_{\gamma}u^\gamma v^i
\]

Then by Bianchi’s second identity (A.2)
\[ T_b^i = \text{Ri}_{\alpha}u^\alpha v^i - \text{Ri}_{\beta}u^\beta v^i + g^\mu v_i^{\prime\prime} \text{Ri}_{\gamma}u^\gamma v^i = \text{Ri}_{\alpha}u^\alpha v^i - \text{Ri}_{\beta}u^\beta v^i + \text{Ri}_{\gamma}u^\gamma v^i = 0 \]

7. 2 dimensional case

Since one of the main motivations for studying flows on manifolds comes from atmospheric models it is interesting to see this case in more detail. Moreover one has to take into account the Coriolis effect. Let us start, however, with the arbitrary 2 dimensional manifold. The main simplification comes from the fact that in this case \( \text{Ri} = \kappa g \) where \( \kappa \) is the Gaussian curvature. So the system can be written as follows
\[
\begin{align*}
u_u + \nu \Delta u - \mu \Delta u - \mu \kappa u + \text{grad}(p) &= 0 \\
- \Delta p - \text{tr}((\nu u, u) - \mu \kappa u, u) + 2 \mu \kappa (\text{grad}(\kappa), u) &= 0 \quad (10)
\end{align*}
\]

The Killing vector fields now satisfy the condition \( g(\text{grad}(\kappa), u) = 0 \); i.e. orbits defined by \( u \) are on the level sets of the curvature. This makes intuitively clear the classical result about existence of Killing fields. Namely locally on 2 dimensional manifolds the space of Killing fields is either three, one or zero dimensional. If \( \kappa \) is constant then we have the three dimensional case. If not the only solution candidates are vector fields which satisfy \( g(\text{grad}(\kappa), u) = 0 \). However, this is only necessary condition so depending on \( \kappa \) the space can be zero or one dimensional. Note that globally the space of Killing fields can be two dimensional as the flat torus shows.

Since the vorticity is important in most of the fluid problems let us examine how it is in our context. Recall that the vorticity is \( \zeta = \text{rot}(u) \) and using the formulas in Appendix B we can write
\[ \zeta = \text{rot}(u) = \text{div}(Ku) = v_i^\prime u_j^i \]

Theorem 7.1. If \( u \) is the solution of (10) then
\[ \zeta_i - \mu \Delta \zeta + g(\text{grad}(\zeta), u) - 2 \mu g(\text{grad}(\kappa), u) = 0 \] (11)

Proof. In 2 dimensional case
\[ Ku = \Delta u + \text{grad}(\text{div}(u)) + \mu u \]

and by the definition of rot it follows that \( \text{rot} \circ \text{grad} = 0 \). Now
\[ \text{rot}(\kappa u) = \text{div}(\kappa Ku) = \kappa u + g(\text{grad}(\kappa), Ku) \]
Then we compute
\[
\rho(\Delta u) = g^g \rho u_{ij}^g \\
\Delta \zeta = \Delta (\zeta^g u_{ij}^g) = g^g \rho u_{ij}^g
\]
Now using the formulas (A.5) and (A.4) we obtain
\[
\Delta \zeta = g^g \rho u_{ij}^g = g^g \rho u_{ij}^g - \kappa u_{ij}^g - \kappa u_{ij}^g
\]
\[
= \rho(\Delta \zeta) - \kappa \zeta - \rho(\kappa \zeta)
\]
Then using the Ricci identity and the formula (A.5) we get
\[
\rho(\nabla u) = \varepsilon_{ij} u_{ij} + \varepsilon_{ij} u_{ij}^g = \varepsilon_{ij} u_{ij}^g + \varepsilon_{ij} \rho(u_{ij}^g - u^g \rho_{ij})
\]
\[
= \varepsilon_{ij} u_{ij}^g + \varepsilon_{ij} u_{ij} + \varepsilon_{ij} (u_{ij} - \kappa u_{ij}^g \varepsilon_{ij}) = \varepsilon_{ij} \rho(u_{ij}^g)
\]
But \(\varepsilon_{ij} \rho(u_{ij}^g) = 0\) and a direct computation shows that \(\varepsilon_{ij} u_{ij}^g = \varepsilon_{ij} u_{ij}^g = \zeta \div (u)\). □

Since the sphere is an important special case let us analyze this case more closely. Let us first recall the following result:

**Theorem 7.2.** Let \(\zeta\) be the solution of (11) on the sphere; then
\[
\frac{1}{2} \mu \|\zeta\|^2 \leq 0
\]

**Proof.** First
\[
\frac{1}{2} \mu \|\zeta\|^2 = \mu \int_M \zeta \Delta \zeta \omega_M - \int_M g(\text{grad}(\zeta), u) \zeta \omega_M + 2 \mu \int_M \kappa \zeta^2 \omega_M
\]
The formula
\[
\text{div}(\zeta \zeta^2 u) = \frac{1}{2} \zeta^2 \text{div}(u) + g(\text{grad}(\zeta), u) \zeta
\]
implies that \(\int_M g(\text{grad}(\zeta), u) \zeta \omega_M = 0\). The result then follows from the inequality (12) because \(\int_M \zeta \omega_M = 0\) and on the sphere \(\lambda_1 = 2 \kappa\). □

Let us again use the decomposition \(u = u^K + u^1\) for the solution of (2) and let \(\zeta = \zeta^K + \zeta^1\) be the corresponding decomposition for the vorticity. Note that for all 2 dimensional manifolds we have
\[
\rho(\text{rot}(u^K)) = 2 \kappa u^K
\]
Multiplying by \(K\) and taking the inner product with \(u^K\) gives
\[
g(\text{grad}(\zeta^K), u^K) = -2 \kappa g(Ku^K, u^K) = 0
\]
On the other hand applying the operator \(\text{rot}\) implies that on the sphere
\[
-\Delta \zeta^K = 2 \kappa \zeta^K
\]
Hence the functions \(\zeta^K\) are actually the first spherical harmonics, i.e. the eigenfunctions of \(-\Delta\) corresponding to the smallest positive eigenvalue.

Interestingly the orthogonality of \(u^K\) and \(u^1\) “descends” to the orthogonality of vorticities.

**Lemma 7.3.** With the above notations on the sphere we have \((\zeta^K, \zeta^1) = 0\).

**Proof.** We compute
\[
\int_M \zeta^K \zeta^1 \omega_M = \int_M \rho(\text{rot}(u^K)) \rho(\text{rot}(u^1)) \omega_M = -\int_M g(Ku^K, \text{grad}(\text{rot}(u^K))) \omega_M
\]
\[
= -\int_M g(u^K, \text{rot}(\text{rot}(u^K))) \omega_M = -2 \kappa \int_M g(u^K, u^K) \omega_M = 0 \quad \square
So on the sphere the dynamics of $\zeta$ happens on the component $\zeta^+$. Then let us see what is the equation for $\zeta^+$. Substituting $u = u^+ + u^-$ and $\zeta = \zeta^+ + \zeta^-$ to (11) and taking into account that (i) $-\Delta \zeta^+ = 2K \zeta^+$, (ii) $g(\nabla(\zeta^+), u^+) = 0$ and (iii) $g(\nabla(\zeta^+), u^-) = 2K g(u^+, Ku^-)$ gives

$$\zeta_t^+ - \mu \Delta \zeta^+ = 2K g(u^+, Ku^-) + g(\nabla(\zeta^+), u^+) + g(\nabla(\zeta^-), u^-) = 2\mu K \zeta^+ = 0$$

This allows us to estimate more precisely the norm of $\zeta^+$. To this end we need the following

**Lemma 7.4.** For any vector fields $u$ and $v$ on a 2 dimensional manifold we have

$$\nabla u v + \nabla v u = \text{div}(v) Ku + \text{div}(Ku) u$$

In particular

$$\nabla u Ku - \text{div}(Ku) Ku = \nabla u - \text{div}(u) u$$

**Proof.** First we have

$$\begin{align*}
\nabla u v &= g^b_{hk}u^{hk}u^b = \varepsilon_{12}(g^{h1}u^2 - g^{h2}u^1) \\
\nabla v u &= g^b_{hk}v^{hk}u^b = \varepsilon_{12}(g^{h1}u^2 - g^{h2}u^1) \\
\nabla u Ku &= g^b_{hk}u^{hk}Ku^b = \varepsilon_{12}(g^{h1}u^2 - g^{h2}u^1) \\
\text{div}(Ku) &= u^bK_{hb} = \varepsilon_{12}(g^{h1}u^2 - g^{h2}u^1)
\end{align*}$$

We have to show that

$$w^b = (g^{h1}u^2 - g^{h2}u^1)v_{hk}^b + (g^{h1}v^2 - g^{h2}v^1)u_{hk}^b = 0$$

But simply expanding the components we can check that $w^b = 0$. □

Note that now we have shown that $\zeta^+$ is orthogonal to the zeroth and first eigenspaces of $-\Delta$. But then by the minimum characterization of the eigenvalues this implies that

$$\frac{\lambda_2}{2} \int_M (\zeta^+)^2 o_M = \frac{\mu}{6} \int_M (\zeta^+)^2 o_M \leq \int_M g(\nabla(\zeta^+), \nabla(\zeta^+)) o_M$$

(13)

**Theorem 7.5.** Let $\zeta^+$ be the solution of (11) on the sphere; then

$$\|\zeta^+\|^2 \leq Ce^{-\mu t}$$

**Proof.** First

$$\begin{align*}
\frac{\lambda_2}{2} \int_M (\zeta^+)^2 o_M &= \mu \int_M \zeta^+ \Delta \zeta^+ o_M - 2\mu \int_M g(u^+, Ku^+)^2 o_M \\
&\quad - \int_M g(\nabla(\zeta^+), u^+) \zeta^+ o_M + \int_M g(\nabla(\zeta^+), u^+) \zeta^+ o_M + 2\mu K \int_M (\zeta^+)^2 o_M
\end{align*}$$

Then we have

$$\int_M g(\nabla(\zeta^+), u^+) \zeta^+ o_M = \int_M g(\nabla(\zeta^+), u^+) \zeta^+ o_M = 0$$

by the same argument as in the proof of **Theorem 7.2**. Then we compute

$$\text{div}(g(u^+, Ku^+)) = g(u^+, Ku^+) + g(\nabla u^+, Ku^+) + g(\nabla u^+, Ku^+) = 2g(u^+, Ku^+) + g(\nabla u^+, u^+)$$

where the second equality holds because $u^+$ is Killing and **Lemma 7.4**. Hence

$$\int_M g(u^+, Ku^+) o_M = 0$$

by **Lemma 3.3**. Then the inequality (13) gives

$$\begin{align*}
\frac{\lambda_2}{2} \int_M (\zeta^+)^2 o_M &= -\mu \int_M g(\nabla(\zeta^+), \nabla(\zeta^+)) o_M - 2\mu K \int_M (\zeta^+)^2 o_M \\
&\leq -8\mu K \int_M (\zeta^+)^2 o_M \quad \Box
\end{align*}$$
8. Coriolis

Let us consider the Coriolis effect. For the simplicity of notation let us assume that our manifold is now the unit sphere $S^2$. The rotation of the sphere has two effects: centrifugal force and Coriolis force. Since the centrifugal force is conservative one can absorb it to the pressure. This modified pressure is still denoted by $p$. From the Coriolis force there comes a new term to the system which is of the form $a Ku$ where $a$ is some function. Let us indicate how to express $a$ in spherical coordinates $(\theta, \phi)$ where $\theta$ is the longitude and $\phi$ is the colatitude. Let us interpret $S^2$ as a submanifold of $\mathbb{R}^3$.

If we now choose $x_3$ axis to be the axis of rotation with the rotation vector $(0, 0, \omega)$ then $a = 2\omega \cos(\phi)$. The system can thus be written as

\begin{align}
    u_t + \nabla_u u - \mu Lu + a Ku + \text{grad}(p) &= 0 \\
    -\Delta p - \text{tr}(\nabla u^2) - g(u, u) - \text{div}(a Ku) &= 0 \\
    \text{div}(u) &= 0
\end{align}

(14)

Since $g(Ku, u) = 0$ the Coriolis term has no effect on the norm: we still have

$$
\frac{d}{dt} \|u\|^2 = -\mu \int_M g(S_u, S_u) \omega_M
$$

However, not all Killing fields are now solutions. Let $u$ be Killing; if it is a solution to (14) then we should have

$$
\text{rot}(auK) = -\mu \text{div}(u) - g(\text{grad}(a), u) = -g(\text{grad}(a), u) = 0
$$

Hence in spherical coordinates $u = c \hat{a}_\theta$ where $c$ is constant. Simple computations show that the corresponding pressure is

$$p_K = \frac{1}{2} \left(c^2 \sin(\phi)^2 + c \omega \cos(2\phi)\right)
$$

Let us thus denote by $(u^K, p_K)$ this solution which in spherical coordinates are given by above formulas. Then we can also in this situation try to look for solutions of the form $u = u^d + \hat{u}$ and $p = p_K + \hat{p}$. Note that here we do not have the result like Theorem 4.1. In spite of this it turns out that the energy of $\hat{u}$ decreases monotonically.

**Theorem 8.1.** Let $u = u^K + \hat{u}$, $p = p_K + \hat{p}$ be a solution to (14). Then

$$
\frac{d}{dt} \|\hat{u}\|^2 \leq 0
$$

**Proof.** Computing as before we find the following system for $(\hat{u}, \hat{p})$.

$$
\hat{u}_t + \nabla_u u^d + \nabla_u \hat{u} + \nabla_\hat{u} \hat{u} - \mu L \hat{u} + \text{grad}(\hat{p}) = 0 \\
\text{div}(\hat{u}) = 0
$$

Then the variational formulation can be written as

$$
\frac{d}{dt} \|\hat{u}\|^2 = -\mu \int_M g(S_{\hat{u}}, S_{\hat{u}}) \omega_M - \int_M g(\nabla_u \hat{u}, \hat{u}) \omega_M - \int_M g(\nabla_\hat{u} \hat{u}, \hat{u}) \omega_M + \mu \int_M g(\hat{L}, \hat{u}) \omega_M - \int_M a g(K \hat{u}, \hat{u}) \omega_M - \int_M g(\text{grad}(\hat{p}), \hat{u}) \omega_M
$$

Then applying Lemma 3.2, Lemma 3.4, divergence theorem and since $g(Ku, u) = 0$, we get

$$
\frac{d}{dt} \|\hat{u}\|^2 = -\mu \int_M g(S_{\hat{u}}, S_{\hat{u}}) \omega_M
$$

Hence the Coriolis term tends to align the flow along the circles of latitude. Note finally that this conclusion depends on the choice of the diffusion operator. If the Hodge Laplacian is used then the solutions simply approach zero because on the sphere there are no harmonic vector fields.

**Appendix A. Notation and some formulas**

Let us review some basic notions of Riemannian geometry. For details we refer to [10] and [13]. For curvature tensor and Ricci tensor there are several different conventions regarding the indices and signs. We will follow the conventions in [10].

The curvature tensor is denoted by $R$ and Ricci tensor by $\text{Ric}$. In coordinates we have

$$
R_{ijk} = R_{ikj} = g^{il} R_{jlk} \\
R_{ij} = g^{kl} R_{kij} = g^{kl} R_{jik} = g^{kl} R_{jlk}
$$

(A.1)
The scalar curvature is $R_m = R^p_{pj}$. The Bianchi identities are

$$R^i_{jkl} + R^i_{jlk} + R^i_{ljk} = 0$$

$$R_{ijlk} + R_{ikjl} + R_{jilk} = 0$$

(A.2)

For general tensors $A$ of type $(m, n)$ the Ricci identity has the form

$$R^i_{jkl} A^{j-l} = \sum_{q=1}^{m} A^{1-i}_{q-j-1} q_{k-l} - A^{1-j}_{i-q-1} q_{k-l} - \sum_{p=1}^{n} A^{1-j+1-i}_{p-j} q_{k-l} R^p_{jl}$$

(A.3)

The following consequences where Ricci identity is used twice are used in many places

$$u^i_{ijk} = u^1_{ijk} + u^1_{i} R^1_{j} - u^1_{j} R^1_{i} - u^1_{k} R^1_{j}$$

$$u^i_{ijk} = u^1_{ijk} + u^1_{i} R^1_{j} - u^1_{j} R^1_{i} - u^1_{k} R^1_{j}$$

(A.4)

In the two dimensional case the curvature tensor can be written as follows:

$$R_{ijk} = \kappa (g_{jk} g_{il} - g_{jl} g_{ik})$$

$$R_{ijk} = \kappa (g_{jk} g_{il} - g_{jl} g_{ik})$$

(A.5)

Here $\kappa$ is the Gaussian curvature.

Appendix B. Operators rot, Rot and curl

Let us denote by $V = C^\infty(M)$ the space of smooth functions on $M$, let $X(M)$ be the space of vector fields and $\Lambda^k M$ the space of $k$ forms. Let us suppose that $M$ is two dimensional. Then we define the operator rot by requiring that the following diagram commutes.

$$
\begin{array}{ccc}
0 & \rightarrow & V \\
\downarrow \text{grad} & & \downarrow \text{curl} \\
X(M) & \rightarrow & V \\
\rightarrow & & \rightarrow \\
0 & \rightarrow & \Lambda^{1} M
\end{array}
$$

(B.1)

Here $\triangleright$ is the usual map $T_f M \rightarrow T^*_f M$ defined by the Riemannian metric and $*$ is the Hodge operator. To express this in coordinates we first define

$$\varepsilon = \sqrt{\det(g)} (dx_1 \otimes dx_2 - dx_2 \otimes dx_1)$$

Note that $\nabla \varepsilon = 0$. Then it is convenient to introduce the map

$$(Ku)^{i} = g_{ij} \varepsilon^{j} u^{i}$$

(B.2)

Intuitively the operator $K$ rotates the vector field by 90 degrees. Then we can write

$$\text{rot}(u) = \text{div}(Ku) = \varepsilon^{j} u_{j}$$

We will also need the Rot operator which is defined by the following diagram

$$
\begin{array}{ccc}
0 & \rightarrow & V \\
\downarrow \text{rot} & & \downarrow \text{curl} \\
X(M) & \rightarrow & V \\
\rightarrow & & \rightarrow \\
0 & \rightarrow & \Lambda^{1} M
\end{array}
$$

(B.3)

Here $\iota_\nu$ is the interior product. In coordinates we have

$$(\text{Rot}(u)^{i}) = -(K \text{grad}(u))^{i} = -\varepsilon_{j}^{i} g^{ij} u_{j} = -\varepsilon^{i} u_{j}$$

Let us now suppose that $M$ is three dimensional. Then we define the operator curl by requiring that the following diagram commutes.

$$
\begin{array}{ccc}
0 & \rightarrow & V \\
\downarrow \text{grad} & & \downarrow \text{curl} \\
X(M) & \rightarrow & V \\
\rightarrow & & \rightarrow \\
0 & \rightarrow & \Lambda^{1} M
\end{array}
$$

(B.4)
Let us now define
\[ \varepsilon = \sqrt{\det(g)} \left( dx_1 \otimes dx_2 \otimes dx_3 - dx_2 \otimes dx_1 \otimes dx_3 \right. \\
- \left. dx_3 \otimes dx_1 \otimes dx_2 + dx_2 \otimes dx_3 \otimes dx_1 + dx_3 \otimes dx_2 \otimes dx_1 + dx_1 \otimes dx_3 \otimes dx_2 \right) \]
Again \( \nabla \varepsilon = 0 \). Then we can express \( \text{curl} \) in coordinates by the formula
\[ (\text{curl}(u))^k = \varepsilon^{ijk}g_{ij}u^j; \]
We can also define the cross product of two vector fields by
\[ (u \times v)^k = (\varepsilon^{ijkl}u^l v^j) = g^{ik}g_{ij}u^j v^i \]
References

Abstract

The defining equations for Killing vector fields and conformal Killing vector fields are overdetermined systems of PDE. This makes it difficult to solve the systems numerically. We propose an approach which reduces the computation to the solution of a symmetric eigenvalue problem. The eigenvalue problem is then solved by finite element techniques. The formulation itself is valid in any dimension and for arbitrary compact Riemannian manifolds. The numerical results which validate the method are given in two dimensional case.

Keywords Killing vector fields, Conformal Killing vector fields, Finite element methods, Riemannian geometry

1 Introduction

Killing vector fields, whose flows generate the isometries on Riemannian manifolds, are fundamental in differential geometry. They arise also indirectly in the study of geodesics. One way to approach the problem is to consider the geodesic flow on the cotangent bundle. Then one tries to find some quantities which are invariant by this flow. If such invariants can be found then the flow is "integrable", i.e. it allows a more explicit description. Darboux in his classic [5, Chapitre II] studied extensively this problem. Apparently he did not explicitly define Killing vector fields, but it turns out that finding a certain type of invariant to the geodesic flow is the same as finding a Killing field on the manifold.

The Killing fields also appear in continuum mechanics because Killing fields are stationary solutions of incompressible Euler and Navier-Stokes equations. This has potentially important consequences when one studies atmospheric models. Let us consider the 2 dimensional Navier-Stokes equations on the sphere. In the absence of external forces the solution tends to some Killing field and not to zero [14]. This phenomenon cannot be observed in the standard setting because boundary conditions do not allow the existence of Killing fields. Now the numerical methods for Navier-Stokes equations typically can add some dissipation to stabilize the computations. However, in the case of the sphere this can have the unintended consequence of dissipating the underlying Killing field. Hence the energy content of the solution is not correct and this can in turn have
significant effects on the computed solutions. We will explore this aspect more thoroughly in a forthcoming paper.

Of course in more complete models of atmospheric flows there are more equations than just the horizontal flow described by the Navier-Stokes equations. However, this horizontal flow is anyway always an important component of the whole model, and hence its analysis will be helpful in understanding the properties of more complicated systems. For more discussion and analysis of atmospheric flows we refer to [majda].

Not all manifolds have Killing fields; the existence of these fields is analyzed for example in [11]. The conclusion is that Killing fields cannot exist if the Ricci tensor is "too negative", and if they exist the space of Killing fields is finite dimensional. Hence as a PDE system the Killing equations are of finite type, i.e. there are only a finite number of free parameters in the general solution. Actually it is rather difficult in practice to determine if the given metric admits any Killing fields. For two dimensional case there is a classical criterion (given below) but higher dimensional cases are still subject to research [10].

Conformal Killing fields is a certain kind of generalization of Killing fields; in other words Killing fields are also conformally Killing, but there may be fields which are conformally Killing but not Killing. Here also the existence of conformally Killing fields depends on the Ricci tensor, but now the Ricci tensor does not have to be "so positive", which allows the existence of more fields. Again as a PDE system the conformal Killing equations are of finite type except in dimension two. In two dimensional case the conformal Killing equations correspond to the Cauchy Riemann equations, so that locally the solution space is infinite dimensional. However, for compact manifolds without boundary the solution space is still finite dimensional.

Conformal Killing fields (which are not Killing) are in fact quite different from Killing fields as we will observe below. The questions where conformal Killing fields arise are apparently of rather different nature than the problems related to Killing fields. As the name suggests, the conformal Killing fields appear in the studies related to the conformal equivalences of Riemannian manifolds. Also in some questions of relativity theory conformal Killing equations appear [2].

Killing fields and conformal Killing fields can be considered as vector fields or covector fields whichever is more convenient. Below we will consider them as vector fields. The defining condition for these fields has also been generalized for other tensor fields [15]. However, below we will only consider vector fields.

Because Killing equations are of finite type, in principle the whole field is determined by the relevant data at one point. However, numerically it is not obvious how to propagate this initial data in a stable way to the whole manifold to obtain a description of the whole field. In fact we are not aware of any general numerical schemes for computing Killing and conformal Killing fields. In this article we propose a method to compute the Killing and conformal Killing vector fields by reducing the problems to a symmetric eigenvalue problem. Killing and conformal Killing vectors then appear as the eigenspace corresponding to the zero eigenvalue of an elliptic operator. Other eigenvalues are positive, and incidentally one could ask if the fields corresponding to positive eigenvalues have any interesting geometric or physical interpretation.

In the numerical solution of the eigenvalue problem we have used standard finite element techniques, and we have used the program FREEFEM++ [9] in our computations. In the case of Klein bottle the identifications of the coordinate domain boundaries are such that we had to program this case with C++. Our formulation gives a well posed problem in any dimension, but below we will give numerical results only in two dimensional case.

The paper is organized as follows. In section 2 we review the necessary background in Riemannian geometry and functional analysis. In section 3 we recall a few relevant properties of the Killing and conformal Killing equations. In section 4 we formulate our problem as an eigenvalue problem and show that the problem is well posed. In section 5 we give the numerical results in two dimensional case which validate our method.

2 Preliminaries and notation

2.1 Geometry

Let us review some basic facts about Riemannian geometry [8, 12]. Let $M$ be a smooth manifold with or without boundary with Riemannian metric $g$. In coordinates we write the vector field $u$ as $u = u^i$ or $u = u^k \partial_k$ if it is convenient to indicate the particular system of coordinates. The Einstein summation convention is used where appropriate. The covariant derivative of $u$ is given by

$$ \nabla u = u^k_j = a^k_j + \Gamma^k_{ij} u^i $$
The semicolon is used for the covariant derivative and comma for the standard derivative. $\Gamma$ is the Christoffel symbol of the second kind. The usual operators are then given by the formulas
\[
\text{div}(u) = \text{tr} (\nabla u) = u^k_k \\
\text{grad}(f) = g^{ij} f_j \\
\Delta f = \text{div}(\text{grad}(f)) = g^{ij} f_{ij}
\]
The divergence operator can extended to general tensors by the formula
\[
\text{div}(T) = \text{tr} (\nabla T).
\]
The metric $g$ induces an inner product for general tensors. For one forms we can simply write
\[
g(\alpha, \beta) = g^{ij} \alpha_i \beta_j.
\]
In addition for this we need the inner product for tensors of type $(1, 1)$. Let $A^\alpha$ be its adjoint, i.e.
\[
g(Au, v) = g(u, A^* v)
\]
for all vector fields $u$ and $v$. Then the inner product on the fibers can be defined by
\[
g(A, B) = \text{tr}(AB^*) = A^\ell_i g^{ij} B^j_k g_{k\ell} = A^j_k B^k_j \tag{2.1}
\]
The curvature tensor is denoted by $R$ and Ricci tensor by $R_i$. There are several different conventions regarding the indices and signs of these tensors. We will follow the conventions in [12]. In coordinates we have
\[
R_{ijk} = R^l_{ijk} \tag{2.2}
\]
In two dimensional case $R_{i} = \kappa g$ where $\kappa$ is the Gaussian curvature.
Let $\partial M$ be the boundary of $M$. The divergence theorem is valid on Riemannian manifolds in the following form:
\[
\int_M \text{div}(u) \omega_M = \int_{\partial M} g(u, \nu) \omega_{\partial M}
\]
where $\nu$ is the outer unit normal and $\omega_M$ is the volume form induced by the metric (or Riemannian density if $M$ is not orientable) and $\omega_{\partial M}$ is the corresponding volume form or density on the boundary.

2.2 PDE

Let $\alpha$ be a multiindex and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Then a general linear PDE can be written as
\[
Au = \sum_{|\alpha| \leq q} b_\alpha \partial^\alpha u = f
\]
where $b_\alpha$ are some known matrices, not necessarily square.

**Definition 2.1** The principal symbol of the operator $A$ is
\[
\sigma A = \sum_{|\alpha| = q} b_\alpha \xi^\alpha
\]
$A$ is elliptic, if $\sigma A$ is injective for all $\xi$ real and $\xi \neq 0$.

Let us from now on suppose that $\sigma A$ is a square matrix because we will not need the more general case in the sequel. Let us then suppose that our PDE system is defined on some Riemannian manifold $M$. Then $\sigma A$ can be interpreted as a $(1, 1)$ tensor, i.e. it defines a map $T_p M \rightarrow T_p M$. The variables $\xi$ can then be interpreted as components of a one form.

The characteristic polynomial of $\sigma A$ is $p_A(\xi) = \det(\sigma A)$. In this case $A$ is elliptic if $p_A \neq 0$ for all $\xi$ real and $\xi \neq 0$. It is clear that the order of $p_A$ must be even if $A$ is elliptic. It is known that for elliptic boundary value problems the number of boundary conditions must be half the order of the characteristic polynomial.

When considering elliptic boundary value problems one needs to impose appropriate boundary conditions. The relevant criterion is known as Shapiro-Lopatinskij condition [1].

2.3 Functional analysis

We will eventually formulate our problem as a spectral problem so let us recall the relevant theorem which we will need. For details we refer to [6, 8].

3
Let us consider some Riemannian manifold \( M \) and let us define the inner product for vector fields by the formula

\[
\langle u, v \rangle = \int_M g(u, v) \omega_M
\]

This gives the norm \( \|u\|_{L^2} = \sqrt{\langle u, u \rangle} \) and the corresponding space is denoted by \( L^2(M) \). In this way we can define the Sobolev inner product

\[
\langle u, v \rangle_{H^1} = \int_M \left( g(u, v) + g(\nabla u, \nabla v) \right) \omega_M
\]

where \( g(\nabla u, \nabla v) \) is defined by the formula (2.1). This gives the norm \( \|u\|_{H^1} = \sqrt{\langle u, u \rangle_{H^1}} \) and the corresponding Sobolev space is denoted by \( H^1(M) \).

Let \( V \) be a real Hilbert space and let \( a: V \times V \rightarrow \mathbb{R} \) be a continuous and symmetric bilinear map. Let \( H \) be another Hilbert space such that \( V \subset H \) with compact and dense injection. Let us consider the following eigenvalue problem:

find \( \lambda \) and \( u \neq 0 \) such that

\[
a(u, v) = \lambda \langle u, v \rangle_{H^1}, \quad \forall v \in V.
\]

Due to symmetry the eigenvalues are real. We say that \( a \) is coercive if there are two constants \( \alpha > 0 \) and \( \mu \in \mathbb{R} \) such that

\[
a(v, v) + \mu \|v\|_V^2 \geq \alpha \|v\|_V^2, \quad \forall v \in V.
\]

The result we will need is the following.

**Theorem 2.2** Let \( a \) be symmetric, continuous and coercive. Then there are real numbers \( \lambda_k \) and elements \( u_k \in V \) such that

\[
a(u_k, v) = \lambda_k \langle u_k, v \rangle_{H^1}, \quad \forall v \in V
\]

where \( -\mu < \lambda_1 \leq \lambda_2 \leq \ldots \) and \( \lambda_k \to \infty \) when \( k \to \infty \). Moreover all eigenspaces are finite dimensional and they are orthogonal to each other with respect to the inner product of \( H \).

### 3 Killing and conformal Killing vector fields

Let \( u \) be a vector field on \( M \) and let us define the following operators:

\[
Su = g^{ij} u_{,ij}^j + g^{ij} u_{,j}^i \\
Cu = Su - \frac{2}{n} \text{div}(u) g^{ij}
\]

**Definition 3.1** A vector field \( u \) is a Killing vector field if \( Su = 0 \) and a conformal Killing field if \( Cu = 0 \).

Let us first summarize some facts about the existence and non-existence of these fields. For more details we refer to [11, 12, 15].

As a PDE system \( Su = 0 \) is a system of \( \frac{1}{2} n(n+1) \) linear first order equations with \( n \) unknown functions. Differentiating all equations we find that one can actually express all second order derivatives in terms of lower order derivatives:

\[
u_{,jk}^i = -u_{,k}^l R_{lkj}^i
\]

where \( R \) is the curvature tensor. Using the language of formal theory of PDE [13, 17] one can say that by prolonging the system once one gets an involutive form which is of finite type. Hence the dimension of the solution space is at most \( \frac{1}{2} n(n+1) \), and this upper bound is actually attained for \( S^n \) and \( \mathbb{R}^n \).

Also \( Cu = 0 \) is a system of \( \frac{1}{2} n(n+1) \) equations with \( n \) unknowns, but now the dimension 2 is a special case. When \( n = 2 \) there are actually only 2 independent equations and it is easily seen that the resulting system is elliptic.\(^1\) Hence locally the space of conformal Killing fields is infinite dimensional. When \( n > 2 \) one has to prolong twice to see that the the system is of finite type so that in this case the solution space is finite dimensional even locally. The relevant computations are carried out in [13, p. 133]. For compact

\(^1\)In \( \mathbb{R}^2 \) with standard metric one obtains Cauchy Riemann equations.
manifolds without boundary the dimension of the solution space is finite even for \( n = 2 \). The upper bound for the dimension of the solutions space is \( \frac{1}{2} (n + 1)(n + 2) \) for \( n > 2 \) in all cases and the same bound is valid when \( n = 2 \) for compact manifolds without boundary. Again this upper bound is attained for the standard spheres.

The existence of Killing and conformal Killing fields depends on the curvature. If the Ricci tensor is everywhere negative definite then there can be no Killing and conformal Killing fields on the manifolds without boundary. On the spheres Ricci tensor is always positive definite; hence the conditions for the existence are "favorable" and in this sense it is rather "natural" that the upper bound for the dimension is attained for the spheres.

Let us indicate one way of checking if there are any Killing fields on the surface for the given metric. Let us introduce the following covectors:

\[
\beta = \frac{1}{2} \, dg(d\kappa, d\kappa) = \kappa_{i} g^{ij} \kappa_{jk},
\]

\[
\alpha = d\Delta \kappa = g^{ij} \kappa_{ijk},
\]

where \( \kappa \) is the Gaussian curvature. Then there is the following classical criterion [10].

**Lemma 3.1** Let \( \mathcal{M} \) be a two dimensional Riemannian manifold and let \( \kappa \) be the Gaussian curvature.

1. If \( \kappa \) is constant then locally the space of Killing fields is three dimensional
2. If \( \kappa \) is not constant and \( d\kappa \otimes \beta \) and \( d\kappa \otimes \alpha \) are symmetric then locally the space of Killing fields is one dimensional
3. Otherwise there are no Killing fields.

On the other hand if one would like to compute which metrics admit Killing fields then the symmetry conditions above give a system of two nonlinear PDE for the three components of the metric. One equation is of the fourth order and the other is of fifth order. Nonlinearity and high order makes this system difficult to handle even though the system is underdetermined.

### 4 Eigenvalue problem

We will from now on always suppose that \( \mathcal{M} \) is compact. Then let us write \( S_{u} \) and \( C_{u} \) when we consider \( Su \) and \( Cu \) as tensors of type \((1,1)\); pointwise they are thus maps \( T_{p} \mathcal{M} \to T_{p} \mathcal{M} \). One can readily check that \( S_{u} \) is symmetric: i.e. \( g(S_{u} v, w) = g(v, S_{u} w) \) for all \( v \) and \( w \). Obviously then \( C_{u} \) is also symmetric.

Let us then introduce the following bilinear maps:

\[
a_{K} : H^{1}(\mathcal{M}) \times H^{1}(\mathcal{M}) \to \mathbb{R}, \quad a_{K}(u, v) = \frac{1}{2} \int_{\mathcal{M}} g(S_{u}, S_{v}) \omega_{\mathcal{M}}
\]

\[
a_{C} : H^{1}(\mathcal{M}) \times H^{1}(\mathcal{M}) \to \mathbb{R}, \quad a_{C}(u, v) = \frac{1}{2} \int_{\mathcal{M}} g(C_{u}, C_{v}) \omega_{\mathcal{M}}
\]

Then we can formulate the following eigenvalue problems:

(K) Find \( u \in H^{1}(\mathcal{M}) \) and \( \lambda \) such that

\[
a_{K}(u, v) = \lambda \int_{\mathcal{M}} g(u, v) \omega_{\mathcal{M}}
\]

for all \( v \in H^{1}(\mathcal{M}) \).

(CK) Find \( u \in H^{1}(\mathcal{M}) \) and \( \lambda \) such that

\[
a_{C}(u, v) = \lambda \int_{\mathcal{M}} g(u, v) \omega_{\mathcal{M}}
\]

for all \( v \in H^{1}(\mathcal{M}) \).

Now evidently \( a_{K}(u, u) \geq 0 \) and \( a_{C}(u, u) \geq 0 \) for all \( u \), and \( a_{K}(u, u) = 0 \) (resp. \( a_{C}(u, u) = 0 \)) only if \( u \) is Killing (resp. conformally Killing) so that the eigenspace of zero eigenvalue is the space of Killing fields (resp. conformally Killing fields).
It is clear that $a_K$ and $a_C$ are symmetric and continuous, so that in particular $\lambda$ must be real. Then we should show that the maps $a_K$ and $a_C$ are coercive. Now in fact the coercivity of $a_K$ in $\mathbb{R}^n$ is a classical result known as Korn’s inequality. This inequality can also be extended to the Riemannian context [3, 16] and the corresponding coercivity result is also valid for the map $a_C$ [4].

The following Theorem gives the result when the manifold has no boundary. This is not a new result, but we think that the proof is interesting because it is simple and it shows the result for both $a_K$ and $a_C$ in the same way. In the following proof we use several formulas which are computed in [14] to which we refer for details.

**Theorem 4.1** The maps $a_K$ and $a_C$ are coercive, if $M$ has no boundary.

**Proof.** Let us introduce the operators $L_K u = \text{div}(S u)$ and $L_C u = \text{div}(C u)$. Then we compute

$$\text{div}(S u) = \frac{1}{2} g(S_u, S_u) + g(L u, v)$$
$$\text{div}(C u) = \frac{1}{2} g(C_u, C_u) + g(L_C u, v)$$

On the other hand

$$L_K u = \Delta_B u + \text{grad}(\text{div}(u)) + \text{Ri}(u)$$
$$L_C u = \Delta_B u + (1 - \frac{2}{n}) \text{grad}(\text{div}(u)) + \text{Ri}(u)$$

where $\Delta_B u = \text{div}(g^{ij} u_u^i u_u^j)$ is the Bochner Laplacian. Hence on the manifolds without boundary

$$a_K(u, u) = \int_M \left( g(\nabla u, \nabla u) + \text{div}(u)^2 - \text{Ri}(u, u) \right) \omega_M$$
$$\geq \int_M \left( g(\nabla u, \nabla u) + (1 - \frac{2}{n}) \text{div}(u)^2 - \text{Ri}(u, u) \right) \omega_M$$

(4.2)

$$= a_C(u, u) \geq \int_M \left( g(\nabla u, \nabla u) - \text{Ri}(u, u) \right) \omega_M$$

Pointwise $\text{Ri}$ can be interpreted as a linear map $T_p M \to T_p M$. Taking the operator norm at each point we can define $\mu = \max_{p \in M} \|R_i\|$. Since $M$ is compact $\mu$ is finite. Hence

$$a_K(u, u) \geq a_C(u, u) \geq \int_M \left( g(\nabla u, \nabla u) - \mu g(u, u) \right) \omega_M \geq \alpha \|u\|_M^2 - (\mu + \alpha) \|u\|_M^2$$

if $0 < \alpha \leq 1$. □

Note that from the formula (4.2) it follows that if $u$ is Killing then

$$\int_M g(\nabla u, \nabla u) \omega_M = \int_M \text{Ri}(u, u) \omega_M$$

and if $u$ is conformally Killing then

$$\int_M \left( g(\nabla u, \nabla u) + (1 - \frac{2}{n}) \text{div}(u)^2 \right) \omega_M = \int_M \text{Ri}(u, u) \omega_M$$

This shows directly that if $\text{Ri}$ is negative definite there can be no Killing and conformally Killing fields.

When the manifold has a boundary the proof is more difficult. Anyway the following results are valid:

If $M$ is compact with Lipschitz boundary $\partial M$ then $a_K$ is coercive [3, 16] and $a_C$ is coercive for $n > 2$ [4].

Our eigenvalue problems are thus well posed. For numerical purposes it is convenient to express $a_K$ and $a_C$ in a different form. Straightforward computations give the following formulas:

$$a_K(u, v) = \int_M \left( g(\nabla u, \nabla v) + \text{tr}(\nabla u \nabla v) \right) \omega_M$$
$$a_C(u, v) = \int_M \left( g(\nabla u, \nabla v) + \text{tr}(\nabla u \nabla v) - \frac{2}{n} \text{div}(u) \text{div}(v) \right) \omega_M$$

It is perhaps useful to interpret the eigenvalue problems in the classical form. Let $p \in \partial M$ and let $\{ \tau_1, \ldots, \tau_{n-1} \}$ be a basis of $T_p \partial M$ and let $\nu$ be the outer unit normal vector. Using the operators $L$ and $L_C$ introduced in the proof of Theorem 4.1 we can write the eigenvalue problems as follows. Again some details of the required computations can be found in [14].
Find $u$ and $\lambda$ such that
\[
\begin{cases}
-L_K u = \lambda u \\
g(\nabla_{\nu} u, \tau_k) + g(\nabla_{\tau_k} u, \nu) = 0, \quad k = 1, \ldots, n-1 \\
g(\nabla_{\nu} u, \nu) = 0
\end{cases}
\]

Find $u$ and $\lambda$ such that
\[
\begin{cases}
-L_C u = \lambda u \\
g(\nabla_{\nu} u, \tau_k) + g(\nabla_{\tau_k} u, \nu) - \frac{2}{n} \text{div}(u) g(\nu, \nu) = 0, \quad k = 1, \ldots, n-1 \\
\frac{1}{2} g(\nabla_{\nu} u, \nu) - \frac{2}{n} \text{div}(u) g(\nu, \nu) = 0
\end{cases}
\]

Note that if $u$ is Killing (resp. conformally Killing) then it satisfies the boundary conditions of problem (K0) (resp. problem (CK0)). Finally let us note that our operators are in fact elliptic.

Lemma 4.1 Operators $L_K$ and $L_C$ are elliptic and moreover their symbols are symmetric.

Proof. Let us denote the identity map in $T_p M$ by $\text{id}$. From the formula (4.1) it readily follows that
\[
\sigma L_K = g(\xi, \xi) \text{id} + g^{ij} \xi^j \xi^k
\]
Then we compute
\[
g(\sigma L_K u, v) = g(\xi, \xi) g(u, v) + g(u, \sigma L_K v) \\
g(\sigma L_K u, u) = g(\xi, \xi) g(u, u) + (\xi^i u^i)^2
\]
Evidently the same computations prove the statement also for $L_C$. \hfill $\Box$

The well posedness of the eigenvalue problem thus also follows from the ellipticity of the operators $L_K$ and $L_C$. Note that the characteristic polynomials of $L_K$ and $L_C$ are of order $2n$ so that the number of the boundary conditions is correct in problems (K0) and (CK0).

5 Numerical results

5.1 Implementation

We have used standard finite element method in our computations, and almost everything was computed with the software FREEFEM++. An example is shown in Figure 5.1 where on the left there is the initial triangulation and on the right is the adapted triangulation. The metric in this case corresponds to the standard torus which will be considered in the examples below.

We will solve problems (K) and (CK) in three cases: Enneper’s surface, torus and the Klein bottle. In case of Enneper’s surface we have a manifold with boundary and a single coordinate chart so that the problem can be formulated in the standard way in FREEFEM++. The torus is a nontrivial manifold but analytically solving problems on the torus means that we look for the periodic solutions. Numerically this can be taken into account by so called periodic boundary conditions, and these are also implemented in FREEFEM++.

The Klein bottle is a nonorientable surface which cannot be embedded in $\mathbb{R}^3$ but it can be embedded in $\mathbb{R}^4$. Here also one can use a single coordinate domain but now the identifications of the domain boundaries are nonstandard and cannot be done with FREEFEM++. In this case we implemented the appropriate identifications and the assembly of relevant matrices directly with C++.

In all cases, for the numerical integration, we used quadrature formula on a triangle which is exact for polynomials of degree less or equal to five. For more informations about the theory and implementation of quadratures, see [7]. We used FREEFEM++ to visualize the computed solutions.

5.2 Special properties of the two dimensional case

In two dimensional case there is a special relationship between Killing and conformal Killing vector fields which is convenient to know when considering the examples. Let us introduce the tensor
\[
\varepsilon = \sqrt{\det(g)} (dx_1 \otimes dx_2 - dx_2 \otimes dx_1)
\]
Note that $\nabla \varepsilon = 0$. Then let us define the operator $K$ by the formula

$$v = Ku \quad \iff \quad v^k = g^{i\ell} \varepsilon_{ij} u^j$$

(5.1)

Intuitively $K$ rotates the vector field by 90 degrees. Then we have

**Lemma 5.1** Let $u$ be a Killing field. Then $Ku$ is a conformal Killing field.

**Proof.** The Killing equations are

$$g^{11} u_1^1 + g^{12} u_2^1 = 0$$
$$g^{11} u_2^2 + g^{12} u_2^2 + g^{12} u_1^1 + g^{22} u_2^1 = 0$$
$$g^{12} v_1^2 + g^{22} v_2^2 = 0$$

Let $v = Ku$; the conformal Killing equations are

$$g^{11} v_2^2 + g^{22} v_2^2 = 0$$
$$g^{11} v_1^1 + 2g^{12} v_2^1 - g^{11} v_2^1 = 0$$

Now simply substituting the covariant derivatives of $v$ to conformal Killing equations one checks that they are satisfied if $u$ satisfies the Killing equations. □

Note that this result shows that Killing vector fields and conformal Killing vector fields are of completely different nature, at least in two dimensional case. For example on the sphere Killing fields generate rotations so they give rise to Hamiltonian dynamics. Conformal Killing fields on the other hand describe the gradient dynamics.

**A surface of revolution** is a surface in $\mathbb{R}^3$ which has the parametrization

$$\varphi(x) = \begin{pmatrix} c_1(x_1) \cos(x_2) \\ c_1(x_1) \sin(x_2) \\ c_2(x_1) \end{pmatrix}$$

The curve $c(x_1) = (c_1(x_1), c_2(x_1))$ is known as the *profile curve*, the curves on the surface with $x_1$ constant are *parallels* and the curves with $x_2$ constant are *meridians*.

**Lemma 5.2** On the surfaces of revolution vector fields $b \partial x_2$ where $b$ is constant are Killing fields. There are no other Killing fields unless the profile curve has a constant curvature.
Proof. The Killing equations for the surfaces of revolution are
\[ |c|^2 u_1^1 + |c| \cdot c'' u_1^3 = 0 \]
\[ |c|^2 u_2^1 + c_1^1 u_2^2 = 0 \]
\[ c_1 u_2^2 + c_1^1 u_1^3 = 0 \]
Clearly the fields \( u = b \partial_{x_2} \) are solutions and by Lemma 3.1 there can be no other Killing fields, unless the profile curve has a constant curvature.

By Lemma 5.1 we thus have the following conformal Killing field on the surface of revolution:
\[ v = K \partial_{x_2} = g^{11} e^{12} \partial_{x_1} \]
(5.2)

5.3 Enneper’s surface

Our first example is the classical Enneper’s surface which is also a minimal surface. Enneper’s surface in \( \mathbb{R}^3 \) is given by the following map:
\[ \varphi(x) = \left( \frac{x_1 - \frac{1}{3} x_2^3}{x_1^2 + x_2^2}, x_2 \right) \]
Let us recall that a coordinate system of a two dimensional Riemannian manifold is isothermal, if the metric is of the form
\[ g = e^{\lambda(x)} (dx_1 \otimes dx_1 + dx_2 \otimes dx_2) \]
for some function \( \lambda \). The metric for Enneper’s surface is of this form with \( \lambda = 2 \ln (1 + |x|^2) \). Simply doing the computations we find that when the parametrization is isothermal then
\[ Su = 0 \iff \begin{cases} u_1^1 \lambda_1 + u_2^1 \lambda_2 + 2u_2^2 = 0 \\ u_1^2 + u_2^2 = 0 \\ u_1^1 - u_2^2 = 0 \end{cases} \]
(5.3)
where comma denotes the standard (not covariant) derivative. Note that the second and third equations are the Cauchy Riemann equations for components of \( u \).

In this case the Killing field can be explicitly computed.

Lemma 5.3 Vector fields \( u = -b x_2 \partial_{x_1} + b x_1 \partial_{x_2} \) where \( b \) is a constant are Killing fields on Enneper’s surface.

Since the curvature is not constant (it is in fact \( \kappa = -4/(1 + |x|^2)^4 \)) there are no other Killing fields by Lemma 3.1.

Proof. The system (5.3) gives in this case
\[ \begin{cases} 2x_1 u_1^1 + 2x_2 u_2^2 + u_2^2 (1 + |x|^2) = 0 \\ u_1^2 + u_3^2 = 0 \\ u_1^1 - u_2^2 = 0 \end{cases} \]
This is equivalent to
\[ \begin{cases} x_1 u_1^1 + x_2 u_2^2 = 0 \\ u_2^2 = 0 \\ x_1 u_1^1 - u_2^2 = 0 \end{cases} \]
From this the result easily follows.

Let us consider the coordinate domain \( \mathcal{D} \) defined by the following boundaries:
\begin{align*}
\partial \mathcal{D}_1 &= \{( \frac{1}{2} (\cos(t) + 1), \sin(t)) \mid t \in [0, \frac{\pi}{2}] \} \\
\partial \mathcal{D}_2 &= \{( \frac{1}{2} - t, 1) \mid t \in [0, \frac{\pi}{2}] \} \\
\partial \mathcal{D}_3 &= \{( \cos(t), \sin(t)) \mid t \in [\frac{\pi}{2}, \pi] \} \\
\partial \mathcal{D}_4 &= \{(- \frac{1}{2} (\cos(t) + 1), - \sin(t)) \mid t \in [0, \frac{\pi}{2}] \} \\
\partial \mathcal{D}_5 &= \{(t, -1) \mid t \in [-\frac{1}{2}, 0] \} \\
\partial \mathcal{D}_6 &= \{(- \frac{1}{2} (\cos(t) + 1), \sin(t)) \mid t \in [3\frac{\pi}{2}, 2\pi] \}
\end{align*}
The triangulated domain, with around 2000 triangles, is shown in Figure 5.2 on the left. The metric was used to adapt the triangulation so that it is quasiform on the surface. The surface with the triangulation is shown in Figure 5.2 on the right.

The command \texttt{rifsimp} in \texttt{Maple} is useful here.
Figure 5.2: The coordinate domain of Enneper’s surface on the left and the corresponding embedding in $\mathbb{R}^3$ on the right with adapted triangulation.

The computation of $a_K$ is in fact easy for all isothermal surfaces, and for Enneper’s surface we obtain:

$$a_K(u,v) = \int_D \left( 2u_1^1v_1^1 + 2u_2^2v_2^2 + u_1^1v_2^2 + u_2^2v_1^1 + u_1^1v_1^1 + u_2^2v_2^2 \right) \left(1 + |x|^2\right)^2 dx_1 dx_2$$

Now in fact we get essentially an exact solution up to rounding errors with $P_1$ elements. Checking the formulas for covariant derivatives one notices that if the components of $u$ and $v$ are polynomials of degree $m$ then the integrand is a polynomial of degree $2m + 2$. Hence using $P_1$ elements integrands are of degree 4 and they are integrated exactly by the default method of FREEFEM++. On the other hand analytically the components of exact solution are polynomials of degree one so that the approximation error is zero [7].

So already with about 100 triangles the approximate eigenvalue is $\lambda \approx 10^{-16}$ and the relative error in $L^2$ norm is about $10^{-13}$ and in $H^1$ norm it is about $10^{-12}$. The computed field is shown in Figure 5.3.

5.4 Torus

The flat torus is isothermal with $\lambda = 0$ so that in this case the Killing fields are $u = b_1 \partial_{x_1} + b_2 \partial_{x_2}$. Hence in particular globally the space of Killing fields can be two dimensional although locally this is impossible by Lemma 3.1. In this case one would also obtain exact solutions up to the rounding errors for the same reason as in the case of Enneper’s surface.

Let us then consider the ”standard” torus, with its Riemannian metric defined by the embedding in $\mathbb{R}^3$. This is a surface of revolution and as a profile curve we can choose

$$c(x_1) = (2 + \cos(x_1), \sin(x_1))$$

The corresponding metric is given by

$$g = dx_1 \otimes dx_1 + (2 + \cos(x_1))^2 dx_2 \otimes dx_2$$

By Lemma 5.2 $u = \partial_{x_2}$ is a Killing field and by formula 5.2

$$v = Ku = K\partial_{x_2} = (2 + \cos(x_1))\partial_{x_1}$$

is a conformal Killing field.

Note that a priori on a general surface of revolution there could be also other conformal Killing fields, but in this particular case one can check that there are in fact no other conformal Killing fields.
Our coordinate domain is thus the square $[0, 2\pi] \times [0, 2\pi]$, with the boundaries appropriately identified. A representative solution for the conformal case, computed with around 2000 triangles, is shown in Figure 5.4. The eigenspace corresponding to the zero eigenvalue is thus two dimensional, and it is spanned by a Killing field and a conformal Killing field which is not Killing. Numerically of course we have two eigenvalues very close to zero and each other. Quantitative results are given below.
5.5 Klein bottle

Let us finally consider the Klein bottle to see that our method works also on nonorientable surfaces. Klein bottle can be embedded in $\mathbb{R}^4$ and one popular parametrization is

$$\varphi(x) = \begin{pmatrix}
(2 + \cos(x_1)) \cos(x_2) \\
(2 + \cos(x_1)) \sin(x_2) \\
\sin(x_1) \cos(x_2/2) \\
\sin(x_1) \sin(x_2/2)
\end{pmatrix} \quad (5.4)$$

The parameter domain is again $[0, 2\pi] \times [0, 2\pi]$ and the sides are identified as in Figure 5.5 on the left. The metric is

$$g = dx_1 \otimes dx_1 + \frac{1}{4} (3 \cos^2(x_1) + 16 \cos(x_1) + 17) dx_2 \otimes dx_2$$

Locally this is like a surface of revolution which looks like (i.e. is isometric to) the surface shown in Figure 5.5 on the right.

Let $a = 3 \cos^2(x_1) + 16 \cos(x_1) + 17$ and $b = \sin(x_1)(3 \cos(x_1) + 8)$. The Killing equations are now

$$u^1_1 = 0$$

$$4u^3_2 + au^1_2 = 0$$

$$au^2_2 - bu^1_2 = 0$$

and it is straightforward to check that $u = \partial_{x_2}$ is a solution. Then by Lemma 5.1 $v = Ku = -\sqrt{\frac{1}{4} (3 \cos^2(x_1) + 16 \cos(x_1) + 17) \partial_{x_1}}$ is a conformal Killing field.

The numerical results are discussed below.

5.6 Numerical errors

As explained above the case of Enneper’s surface is rather special so let us here consider only the torus and the Klein bottle in more detail. We used $P^1_1$ elements for the Klein bottle and $P^2_2$ elements for the torus. In both cases we computed the solutions to problems (K) and (CK) with several triangulations. As we can see in tables 2 and 3, even with few triangles (around 100), we have already quite a good approximation. In the conformal Killing case the eigenspace corresponding to zero eigenvalue is two dimensional. Numerically we have two eigenvalues very close to zero. Note that the approximation to the Killing field is much better than the approximation to the conformal Killing field. This is because the components of the Killing field are simply constants so that the approximation error is zero and we see only the error arising in the numerical integration.

For completeness we computed the order of convergence in a standard way, i.e. we computed $k$ such that

$$\varepsilon \approx Ch^k$$

where $h$ is the maximum length of the triangulated domain and $\varepsilon$ is the error. With 40 different triangulations for the example of the standard torus (5.4), results are presented in the table 1. Results for the conformal
\[ \varepsilon \equiv |\lambda_h - \lambda| \quad \varepsilon = \|u_h - u\|_{L^2} \quad \varepsilon = \|u_h - u\|_{H^1} \]

| KF | $\varepsilon \equiv |\lambda_h - \lambda|$ | $\varepsilon = \|u_h - u\|_{L^2}$ | $\varepsilon = \|u_h - u\|_{H^1}$ |
|----|--------------------------------|----------------|----------------|
| k  | 8.8                           | 6.55           | 3.45           |

| CKF | $\varepsilon \equiv |\lambda_h - \lambda|$ | $\varepsilon = \|u_h - u\|_{L^2}$ | $\varepsilon = \|u_h - u\|_{H^1}$ |
|-----|--------------------------------|----------------|----------------|
| k   | 4.25                          | 3.81           | 2.67           |

Table 1: The estimated order of convergence.

<table>
<thead>
<tr>
<th>100 triangles</th>
<th>With adaptation</th>
<th>Without adaptation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalue</td>
<td>$10^{-7}$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>$L^2$ norm of error</td>
<td>$10^{-8}$</td>
<td>$10^{-9}$</td>
</tr>
<tr>
<td>$H^1$ norm of error</td>
<td>$10^{-7}$</td>
<td>$10^{-4}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>2 000 triangles</th>
<th>With adaptation</th>
<th>Without adaptation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalue</td>
<td>$10^{-10}$</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>$L^2$ norm of error</td>
<td>$10^{-14}$</td>
<td>$10^{-10}$</td>
</tr>
<tr>
<td>$H^1$ norm of error</td>
<td>$10^{-14}$</td>
<td>$10^{-8}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Klein</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalue</td>
<td>$10^{-7}$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>$L^2$ norm of error</td>
<td>$10^{-6}$</td>
<td>$10^{-4}$</td>
</tr>
<tr>
<td>$H^1$ norm of error</td>
<td>$10^{-5}$</td>
<td>$10^{-3}$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>100 triangles</th>
<th>With adaptation</th>
<th>Without adaptation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalue</td>
<td>$10^{-8}$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>$L^2$ norm of error</td>
<td>$10^{-8}$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>$H^1$ norm of error</td>
<td>$10^{-7}$</td>
<td>$10^{-5}$</td>
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<table>
<thead>
<tr>
<th>2 000 triangles</th>
<th>With adaptation</th>
<th>Without adaptation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eigenvalue</td>
<td>$10^{-11}$</td>
<td>$10^{-7}$</td>
</tr>
<tr>
<td>$L^2$ norm of error</td>
<td>$10^{-11}$</td>
<td>$10^{-6}$</td>
</tr>
<tr>
<td>$H^1$ norm of error</td>
<td>$10^{-14}$</td>
<td>$10^{-8}$</td>
</tr>
</tbody>
</table>

Killing field (which are not Killing) are close to what one expects of $P_2$ elements. For the Killing fields the convergence is faster because we see only the error due to numerical integration. The results are quite similar in the case of the Klein bottle. The convergence for conformal Killing fields are what one expects of $P_1$ elements, and again for Killing fields the order of the convergence is related to the order of numerical integration.
Figure 5.6: Component $u^1$ of a conformal Killing field which is not Killing on the Klein Bottle without (on the left) and with adaptation (on the right) of the metric.

The adaptation of the metric is important for computations as shown in Figure 5.6. It represents the first component of the conformal Killing field which is not Killing on the Klein bottle. On the left, the domain is triangulated without adaptation, and it shows that the solution is deformed. That is not the case with the same number of triangles using an adapted mesh (on the right). It implies that $L^2$ and $H^1$ errors can increase significantly without an adapted mesh (see tables 2 and 3).

References


III
On some classes of Riemannian manifolds

Maryam Samavaki
Department of Physics and Mathematics
University of Eastern Finland
P.O. Box 111, FI-80101 Joensuu, Finland
maryam.samavaki@uef.fi

Jukka Tuomela
Department of Physics and Mathematics
University of Eastern Finland
P.O. Box 111, FI-80101 Joensuu, Finland
jukka.tuomela@uef.fi

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Abstract

We study several classes of Riemannian manifolds which are defined by imposing a certain condition on the Ricci tensor. We consider the following cases: Ricci recurrent, Cotton, quasi Einstein and pseudo Ricci symmetric condition. Such conditions can be interpreted as overdetermined PDE systems whose unknowns are the components of the Riemannian metric, and perhaps in addition some auxiliary functions. Hence even if the dimension of the manifold is small it is not easy to compute interesting examples by hand, and indeed very few examples appear in the literature. We will present large families of nontrivial examples of such manifolds. The relevant PDE systems are first transformed to an involutive form. After that in many cases one can actually solve the resulting system explicitly. However, the involutive form itself already gives a lot of information about the possible solutions to the given problem. We will also discuss some relationships between the relevant classes.

Keywords
Cotton tensor, conformally conservative manifold, pseudo Ricci symmetric manifold, quasi Einstein manifold, Ricci recurrent manifold, overdetermined PDE

1 Introduction

In the following we are going to analyze and present examples of several classes of Riemannian manifolds: Ricci recurrent, pseudo Ricci symmetric, Cotton and quasi Einstein manifolds. Sometimes these classes are understood in a generalized sense without requiring the positive definiteness of the metric. In the following we will on the other hand only consider the positive definite case. However, in the actual construction of examples the positive definiteness does not play a role, so that using the same approach one can also construct examples which are not positive definite.

Ricci recurrent manifolds were first considered in [12] (even earlier recurrent Riemannian manifolds were introduced in [18]). Since then this class has been analyzed by many authors, see for example [7] and references therein, where also various generalizations and extensions of this notion are considered.

The concept of pseudo Ricci symmetric manifold was perhaps explicitly first introduced in [3]. However, earlier in [20] authors had obtained the characterizing condition when analyzing the existence of another structure on Riemannian manifolds.
As a term the Cotton manifold or Cotton metric is not very common; it is used in [10]. However, the Cotton tensor, introduced in [5], which gives the defining condition for such manifolds appears in wide variety of questions. We will explain below how Cotton manifolds are related to other classes of Riemannian manifolds.

There are in fact at least 3 different definitions for quasi Einstein manifolds. The one we are interested in was apparently introduced in [1]. The quasi Einstein property was then a special case of larger class of Riemannian manifolds. Other definitions, neither directly related to the present article nor to each other can be found for example in [4] and [2]. As the name suggest these spaces are typically related to problems in general relativity and the hence the metric in that case is typically not positive definite.

We will analyze some connections of the above classes of manifolds. However, the main part of our paper is devoted to the construction of large families examples of these different spaces. In the papers where these types of Riemannian manifolds are considered there are very few actual examples. In some sense this is natural since producing an example implies solving a relatively big system of PDE. Below we will show how to use the theory of overdetermined PDE (also called formal theory of PDE) [9, 14, 19] to produce solutions. The conclusion is in fact that with appropriate tools it is not particularly hard to find examples.

Below we have chosen examples more or less randomly with no particular application in mind. However, the reader who wants solutions of some specific form can easily adapt our approach to other contexts. Of course this approach does not always lead to explicit solutions, but the analysis can still give important information about the nature of solutions. In fact the special form of the system that is obtained in the analysis is even suitable for numerical computations, if one wants to explore numerically different possibilities.

The paper is organized as follows. In section 2 we recall some notions which are needed in the analysis. In section 3 the classes of Riemannian manifolds are introduced, and the relationships between them are analyzed. Then in section 4 we formulate our computational problems precisely. Finally in section 5 we present and discuss the examples and in section 6 there are some concluding remarks.

2 Preliminaries

2.1 Geometry

Let $M$ be a smooth $n$ dimensional manifold with Riemannian metric $g$. The pointwise norm of a tensor $T$ is denoted by $|T|$. The covariant derivative is denoted by $\nabla$. We say that a tensor $T$ is parallel, if $\nabla T = 0$. The curvature tensor is denoted by $R$ and the Ricci tensor is $R_{ij} = R^{k}_{jk}$. There are several conventions regarding the signs and indices of curvature tensors. We will follow [13].

In several places we will need the Ricci identity which for general tensors $A$ of type $(m,n)$ has the form

$$A_{\ell_1 \cdots \ell_m}^{i_1 \cdots i_n} \cdot A_{i_1 \cdots i_n}^{\ell_1 \cdots \ell_m} = \sum_{q=1}^{n} A_{\ell_1 \cdots \ell_q \cdots \ell_m}^{i_1 \cdots i_q \cdots i_n} R^{\ell_q}_{\ell_1 \cdots \ell_q} - \sum_{p=1}^{m} A_{i_1 \cdots i_{p-1} \ell_{p+1} \cdots \ell_m}^{\ell_1 \cdots \ell_{p-1} \cdots \ell_m} R^{\ell_p}_{i_1 \cdots i_{p-1} \ell_{p+1} \cdots \ell_m}.$$  \hspace{1cm} (2.1)

The Bianchi identity is

$$R_{hijk,l} + R_{likt,j} + R_{hi\ell,j,k} = 0 \hspace{1cm} (2.2)$$

By multiplying above equation on $g^{hk}$, we have

$$R_{hij,l} - R_{li,j} = R_{hij,l}$$

which then implies that

$$sc_{,k} = 2 \text{div}(Ri) = 2 R_{hij}$$ \hspace{1cm} (2.3)

Let us then define some classical tensors which are needed in the sequel.

**Definition 2.1** Let $M$ be a $n$ dimensional Riemannian manifold with metric $g$. Then

(i) Schouten tensor is

$$S = Ri - \frac{sc}{2n - 2} g$$

(ii) Cotton tensor is

$$C_{ijk} = S_{ijk} - S_{ikj}$$

2
(iii) Weyl tensor is
\[ W_{hijk} = R_{hijk} + \frac{sc}{(n-1)(n-2)} \left( g_{hk}g_{ij} - g_{hj}g_{ik} \right) - \frac{1}{n-2} \left( R_{hkij} - R_{hjik} + R_{ihjk} - R_{ihkj} \right). \]

In some references Schouten tensor is some constant multiple of \( S \) given above. Note that \( W = 0 \) when \( n \leq 3 \) and \( S = 0 \) when \( n = 2 \). Let us also recall

**Theorem 2.1** Let \( M \) be a \( n \) dimensional Riemannian manifold. Then

1. \( M \) is conformally flat if and only if \( C = 0 \) when \( n = 3 \) or \( W = 0 \) when \( n \geq 4 \).
2. \( \text{div}(W) = \frac{n-3}{n-2} C \) when \( n \geq 4 \).

### 2.2 Determinantal varieties

Let \( \mathbb{R}^{m \times n} \) be the vector space of real \( m \times n \) matrices and let \( V_r \) be the subvariety of matrices of rank at most \( r \). This is a determinantal variety, defined by setting to zero all minors of size \((r+1) \times (r+1)\). There are thus \( \binom{n}{r+1} \) \( \binom{n}{r+1} \) polynomials which generate the ideal defining \( V_r \). However, not all generators are algebraically independent and one can show that

\[ \text{codim}(V_r) = (m-r)(n-r) \]

Let then \( S_n \) be the vector space of real \( n \times n \) symmetric matrices and let \( V^s_r \) be the subvariety of symmetric matrices of rank at most \( r \). The relevant ideal is now generated by \( \binom{n}{r+1}^2 \) polynomials, but due to symmetry we have much less independent generators, and one can show that in this case

\[ \text{codim}(V^s_r) = \left( \frac{n-r+1}{2} \right)^2 \]  \( (2.4) \)

### 2.3 Overdetermined PDE

The existence of various manifolds considered below depends on the solvability of certain systems of overdetermined PDE. For a general overview of overdetermined PDE we refer to [9, 14, 19] and references therein. Both books contain also historical comments on the development of the subject which started at the end of 19th century.

Very early it was realized that before one could actually prove any existence results for overdetermined PDE in some definite function space one should first analyze the structural properties of the system. The main difficulty of the analysis of overdetermined systems is related to integrability conditions: in other words by differentiating the equations one may find new equations which are algebraically independent of the original equations. The process of finding the integrability conditions is called completion, and the goal was to find all integrability conditions.

Analysis of the completion process lead to two complementary approaches: geometric and algebraic. The geometric approach is based on interpreting PDE as submanifolds of jet spaces. The algebraic approach requires that the nonlinearities are polynomial and hence the equations can be interpreted as differential polynomials and the systems themselves are differential ideals generated by the given differential polynomials.

It turns out that proving that the system is complete, or that the system can actually be completed, is quite tricky and we simply refer again to [9, 14, 19] for details. In spite of this heavy machinery which is required for the theory the end result is perhaps surprisingly constructive: there are actual algorithms for computing the completed system, i.e. the system which contains all integrability conditions. The completed system is called the involutive system, and the completion algorithm is usually known as Cartan-Kuranishi algorithm.

The analysis of structural properties of overdetermined PDE is also called formal theory of PDE. The word formal appears because one can say that the involutive form of the system has solutions as formal power series. One can say that in the involutive system all relevant information about the system is explicit while in the initial system it was only implicit.

An analogous situation arises in polynomial algebra. A polynomial system generates an ideal which in turn defines the corresponding variety. Now computing the Gröbner basis of the ideal gives a lot of information about the variety [6]. Intuitively one may think about computing the involutive form of a system of PDE like computing the Gröbner basis of an ideal.
The idea of Gröbner bases can be generalized to differential equations, where equations are interpreted as differential polynomials [11]. However, not all properties of Gröbner bases of the algebraic case carry over to the differential case. Anyway the ideas related to Gröbner bases and ideal theory in general are present in the actual implementations of completion algorithms.

One final comparison to polynomial case is perhaps helpful. In the polynomial case any variety can be decomposed to a finite number of irreducible varieties which means that any polynomial ideal is an intersection of finite number of primary ideals. This property is still valid in the differential context in the following form: any radical differential ideal is a finite intersection of prime differential ideals. Hence if the nonlinearities are polynomial, and they will be in all systems considered below, one may also try to find the decomposition of the involutive form. Evidently finding this decomposition greatly facilitates any further analysis of the system.

In what follows we will use the algorithm rifsimp which is described in detail in [16], see also [15]. The acronym rif means reduced involutive form. This algorithm assumes that the nonlinearities are polynomial, and that the implied differential field is the field of rational functions. It can also handle inequations and it can compute the decomposition of the system.

The algorithm is implemented as the command rifsimp in Maple. In setting up the systems of equations the Differential Geometry package of Maple was also very useful.

Finally we note that the word "overdetermined" is a bit misleading. This term is traditionally used, but actually one simply means the analysis of general PDE systems. The important concept is the involutivity (or some other canonical form), and in many (or even most) cases it is not necessary to define precisely what is meant by the term overdetermined (or underdetermined). In particular below this definition is not needed. Also one can find (at least) two different definitions in the literature which are both reasonable in their ways; see [9] and [19] for these different definitions.

3 Some properties and relationships between various classes

In the following we will consider several classes of Riemannian manifolds. These classes are defined by requiring that the corresponding Ricci tensor satisfies some condition $P$. In this case we can also say that the manifold or the Riemannian metric is of the type $P$. Of course we will always assume that $R_i \neq 0$.

**Definition 3.1** Ricci tensor is

- **Ricci recurrent**, RR, if there is a nonzero one form $\beta$ such that
  \[ R_{i,j,\ell} = \beta_{\ell} R_{i,j} . \]  \hspace{1cm} (3.1)

- **pseudo Ricci symmetric**, PRS, if there is a nonzero one form $\alpha$ such that
  \[ R_{i,j,\ell} = 2\alpha R_{i,j} + \alpha_i R_{\ell j} + \alpha_j R_{i,\ell} \]  \hspace{1cm} (3.2)

- **quasi Einstein**, QE, if there are functions $a$ and $b$, and one form $\omega$ such that
  \[ R_i = a g + b \frac{\omega \otimes \omega}{|\omega|^2} \]  \hspace{1cm} (3.3)

If $b = 0$ the Ricci tensor is Einstein.

- **Cotton**, CO, if the Cotton tensor is zero.

In dimension two we have $R_i = \kappa g$ where $\kappa$ is the Gaussian curvature. Hence any manifold is Einstein, Cotton and Ricci recurrent. On the other hand no manifold is pseudo Ricci symmetric. Hence from now on we suppose that the dimension $n \geq 3$.

It is clear that if $R_i$ is parallel the RR condition cannot be satisfied, and on the other hand the CO condition is always satisfied. Below we will see that PRS condition is incompatible with parallelism.

If the metric satisfies the condition $\text{div}(W) = 0$ it is sometimes said to be *conformally conservative*. Hence by Theorem 2.1 Cotton manifolds are conformally conservative when $n \geq 4$ and conformally flat when $n = 3$.

Finally recall that if $M$ is an Einstein manifold then $a = sc/n = \text{constant}$. 

1https://www.maplesoft.com/
3.1 Ricci recurrent

Let us then start with the RR case. Multiplying (3.1) by $\mathbf{R}^{ij}$ we obtain

\[ \beta = \frac{\mathbf{R}^{ij} \mathbf{R}_{ijkl}}{|\mathbf{R}|^2} = \frac{1}{2} \nabla \ln(|\mathbf{R}|^2) \]

From this we get the following characterization.

**Lemma 3.1** Let $\mathbf{N} \mathbf{R} \mathbf{i} = \frac{\mathbf{R} \mathbf{i}}{|\mathbf{R}|}$. Then $\mathbf{R} \mathbf{i}$ is RR if and only if $\mathbf{N} \mathbf{R} \mathbf{i}$ is parallel.

**Proof.** Simply taking the covariant derivative of $\mathbf{N} \mathbf{R} \mathbf{i}$ we see that it is zero precisely when $\mathbf{R} \mathbf{i}$ is RR with $\beta$ as given above. □

However, it turns out that the RR case can be characterized purely in an algebraic way. In [17] it is shown that actually

\[ \mathbf{R}^{ij} \mathbf{R}_{ijkl} = \frac{1}{2} \mathbf{sc} \mathbf{R}^{ij}_{il} \quad (3.4) \]

Note that this result crucially depends on the fact that $g$ is positive definite. But this leads easily to the following characterization of the Ricci tensor.

**Theorem 3.2** Suppose that $\mathbf{R} \mathbf{i}$ is recurrent. Then it has a double eigenvalue $\frac{\mathbf{sc}}{2}$ and eigenvalue zero of multiplicity $n - 2$. Moreover

\[ \beta = \nabla \ln(\mathbf{sc}) \quad , \quad \mathbf{sc}^2 = 2 |\mathbf{R}|^2 \quad \text{and} \quad \mathbf{R} \mathbf{i} \beta = \frac{\mathbf{sc}}{2} \beta . \]

**Proof.** Let $\lambda_j$ be the eigenvalues of $\mathbf{R} \mathbf{i}$. Then the formula (3.4) implies that

\[ \lambda_j^2 = \frac{1}{2} (\lambda_1 + \cdots + \lambda_n) \lambda_k . \]

This gives the first statement. Since the scalar curvature cannot be zero the formula for $\beta$ is obtained multiplying (3.1) by $g^{ij}$. Taking the trace in (3.4) gives the second formula. Then by formula (2.3) we have

\[ \mathbf{sc} \beta_{\ell} = \mathbf{sc}_{\ell} = 2 g^{hk} \mathbf{R}_{h\ell,k} = 2 g^{hk} \delta_{\ell} \mathbf{R}_{h\ell} = 2 \mathbf{R} \mathbf{i} \beta . \]

From this we immediately get

**Corollary 3.1** Ricci tensor cannot be at the same time RR and CO.

**Proof.** If the Ricci tensor satisfies the condition RR then

\[ \mathbf{S}_{ij,k} = \frac{\mathbf{sc}}{\mathbf{sc}} (\mathbf{R} \mathbf{i} - \frac{1}{2(n-1)} \mathbf{sc} g_{ij}) = \frac{\mathbf{sc}}{\mathbf{sc}} S_{ij} \]

Note that Schouten tensor is also "recurrent" in this case. Now by previous Theorem we know that $\mathbf{R} \mathbf{i}$ has an eigenvector $\mathbf{v}$ such that $\mathbf{R} \mathbf{i} \mathbf{v} = 0$ and $g(\nabla \mathbf{v}, \mathbf{v}) = 0$. This implies that

\[ C_{ijk} \mathbf{v}^l = S_{ijk} \mathbf{v}^l - S_{ikj} \mathbf{v}^l = -\frac{1}{2(n-1)} \mathbf{sc} \mathbf{k} \mathbf{v}^l \neq 0 . \]

3.2 Pseudo Ricci symmetric

It turns out that the associated one form is almost the same in the PRS case.

**Lemma 3.3** Let the Ricci tensor be PRS with associated one form $\alpha$. Then $\mathbf{R} \mathbf{i} \alpha = 0$ and

\[ \alpha = \frac{1}{2} \nabla \ln(|\mathbf{R}|^2) \]

If the scalar curvature is not zero then

\[ \alpha = \frac{1}{2} \nabla \ln(\mathbf{sc}) \quad \text{and} \quad \mathbf{sc}^2 = c |\mathbf{R}|^2 \]

for some constant $c$. 5
It is easy to see that if the scalar curvature is constant then it has to be zero. However, it seems to be an open problem if the case $\text{sc} = 0$ can actually occur. If one does not assume the positive definiteness of the metric then it is easy to construct examples with $\text{sc} = 0$. However, we were unable to find an example with a positive definite metric.

**Proof.** Multiplying (3.2) with $g^{ij}$ and using the formula (2.3) we get

$$\text{sc}_{i,j} = 2g^{ij}\text{R}_{ij,\xi} = 4\alpha_i g^{ij}\text{R}_{ij} + 2\alpha_i \text{sc} = 6\alpha_i \text{R}_{ij} + 2\alpha_i \text{sc}$$

On the other hand multiplying (3.2) with $g^{ij}$ we get

$$\text{sc}_{i,j} = 2\alpha_i \text{sc} + 2\alpha_i \text{R}_{id}$$

Hence $\text{R}\alpha = 0$. The expression for $\alpha$ is then obtained by multiplying (3.2) with $\text{R}^{ij}$. If $\text{sc} \neq 0$ we have $\text{sc}_{i,j} = 2\alpha_i \text{sc}$ which gives the other expression for $\alpha$.

From this we get immediately

**Corollary 3.2**

(i) The Ricci tensor cannot be both $\text{RR}$ and $\text{PRS}$.

(ii) If $\text{R}$ is parallel then the $\text{PRS}$ condition cannot be satisfied.

**Proof.** (i) The previous Lemmas imply that the associated one forms in the two cases satisfy $\beta = 2\alpha$. Hence

$$\beta \text{R}\alpha = \text{R}_{ij,\xi} = 2\alpha_i \text{R}_{ij} + \alpha_j \text{R}_{ij} + \alpha_j \text{R}_{id}$$

implies that $\alpha_i \text{R}_{ij} + \alpha_j \text{R}_{ij} = 0$. Multiplying this with $\alpha^i$ gives $|\alpha|^2 \text{R} = 0$ which is impossible.

(ii) If $\text{R}$ is parallel, then multiplying (3.2) with $\alpha^i$ gives $2|\alpha|^2 \text{R} = 0$ which is impossible. 

### 3.3 Quasi Einstein

Let us then analyze quasi Einstein structure. To this end it is convenient to formulate the condition (3.3) differently. Let us introduce the tensor $T = \text{R}_{ij} - a g_{ij}$. Then we can say

(i) $M$ is an Einstein manifold, if $T = 0$ and $a = \text{sc}/n$.

(ii) $M$ is a quasi Einstein manifold, if the matrix rank of $T$ is one.

In this way we see that the one form $\omega$ and the function $b$ are actually quite irrelevant in the analysis of the existence of quasi Einstein structure.

Now if we want that the symmetric tensor $T$ is of matrix rank one then according to the formula (2.4) there are only $\binom{n}{2}$ algebraically independent differential equations. On the other hand there are $1 + \binom{n+1}{2}$ unknowns, namely $a$ and the components of $g$, so it should not be too difficult to find solutions. Note that no derivatives of $\alpha$ appear in the equations, so it is natural to first eliminate $a$ from the equations and then consider the system obtained for the metric $g$. This is the approach we will follow below.

Of course when we suppose that $g$ is of specific form it is not a priori clear how many independent differential equations are obtained in this way. Note that as a PDE system $\text{QE}$ system is essentially different from RR and PRS systems. $\text{QE}$ is a fully nonlinear system, i.e. nonlinear in highest derivatives while RR and PRS systems are quasilinear.

Once the appropriate $T$ is found $b$ and $\omega$ can easily be computed. By taking traces in the formula (3.3) one sees immediately that $b = \text{sc} - na$ and then $\omega$ can be solved from the linear system

$$T_{ij} \omega^j = (\text{sc} - na) \omega_i$$

Still another way to characterize the $\text{QE}$ case which is actually useful when considering examples is that if $\text{R}$ has a simple eigenvalue $\text{sc} - (n-1)a$ corresponding to the eigenvector $\omega$, and all vectors orthogonal to $\omega$ are eigenvectors with eigenvalue $a$ whose multiplicity is thus $n-1$. But this formulation immediately gives

**Theorem 3.4** If the Ricci tensor is recurrent and $n = 3$ then it is automatically quasi Einstein. If $n > 3$ the Ricci tensor cannot be both recurrent and quasi Einstein.

**Proof.** By Theorem 3.2 the eigenvalue structure of $\text{R}$ can satisfy both RR and $\text{QE}$ conditions only if $n = 3$. In this case if $\omega$ is the eigenvector corresponding to the zero eigenvalue we can write the Ricci tensor as follows

$$\text{R}_{ij} = \frac{\text{sc}}{2} \left( g_{ij} - \frac{\omega_i \omega_j}{|\omega|^2} \right)$$
On the other hand PRS and QE conditions can be satisfied in any dimension. In these cases the form of the Ricci tensor is as follows.

**Theorem 3.5** Let us suppose that both PRS and QE conditions are satisfied and let \( \alpha \) be the associated one form. If \( a \neq 0 \) we have

\[
\mathbf{R}_{ij} = \frac{\text{sc}}{n-1} \left( g_{ij} - \frac{\alpha_i \alpha_j}{|\alpha|^2} \right)
\]

If \( a = 0 \) then

\[
g(\omega, \alpha) = 0, \quad \mathbf{R}_{ij} = \text{sc} \frac{\omega_i \omega_j}{|\omega|^2} \quad \text{and} \quad \nabla \omega = |\alpha|^2 \omega.
\]

**Proof.** By multiplying formula (3.3) with \( \alpha^i \) and applying Lemma 3.3, we have

\[
a \alpha_j + b \frac{g(\omega, \alpha) \omega_j}{|\omega|^2} = 0 \quad (3.5)
\]

If \( a \neq 0 \) and \( g(\omega, \alpha) \neq 0 \) then \( \alpha \) and \( \omega \) are linearly dependent and we may choose \( \omega = \alpha \) which gives

\[
a = \frac{\text{sc}}{n-1} \quad \text{and} \quad b = -a.
\]

When \( a = 0 \) the first two statements are obvious. To get the third we take the covariant derivative of the formula \( \mathbf{R}_{ij} = \text{sc} \frac{\omega_i \omega_j}{|\omega|^2} \), and then multiply it with \( \alpha^i \). Then using the formula (3.2) and simplifying we get the result.

**Lemma 3.6** Suppose that \( \mathbf{R} \) satisfies the PRS condition and that \( \text{sc} \neq 0 \); then

(i) if \( \mathbf{R} \) is also QE with \( a \neq 0 \) then it is CO

(ii) if \( \mathbf{R} \) is also CO then it is QE

**Proof.** If the PRS condition is satisfied and \( \text{sc} \neq 0 \) then the Cotton tensor is

\[
\mathbf{C}_{ijk} = \frac{\text{sc} \lambda_k}{2 \text{sc}} \left( \mathbf{R}_{ij} - \frac{\text{sc}}{n-1} g_{ij} \right) - \frac{\text{sc} \lambda_i}{2 \text{sc}} \left( \mathbf{R}_{ik} - \frac{\text{sc}}{n-1} g_{ik} \right)
\]

Now simply substituting the expression for \( \mathbf{R} \) given in Theorem 3.5 shows that \( \mathbf{C} = 0 \) which proves the statement (i).

On the other hand it is easy to check that the Cotton tensor of the above form can be zero only if the matrix rank of \( \mathbf{R}_{ij} - \frac{\text{sc}}{n-1} g_{ij} \) is one which is precisely the QE condition.

There is a completely different way to construct QE metrics which we now describe. The form of the condition makes one think about conformal equivalence. Let \( g \) be a given metric and let \( \hat{g} = \exp(2\lambda)g \) be a metric that is conformally equivalent to it. If now \( \hat{\mathbf{R}} \) is the Ricci tensor associated to \( \hat{g} \) then

\[
\hat{\mathbf{R}}_{ij} = \mathbf{R}_{ij} - (n-2) (\lambda_{ij} - \lambda_i \lambda_j) - (\Delta \lambda + (n-2) \nabla \lambda)^2 g_{ij}
\]

Hence if we can find a metric \( g \) and a function \( \lambda \) such that \( \mathbf{R}_{ij} = (n-2) \lambda_{ij} \) then \( \hat{\mathbf{R}} \) is quasi Einstein:

\[
\hat{R}_{ij} = -(\Delta \lambda + (n-2) \nabla \lambda)^2 \exp(2\lambda) \hat{g}_{ij} + (n-2) \lambda_i \lambda_j
\]

Note that the equations are trivially satisfied if \( \hat{\mathbf{R}} = \nabla \nabla \lambda = 0 \). However, if \( \nabla \lambda \neq 0 \) there is still a nontrivial solution

\[
\hat{\mathbf{R}}_{ij} = -(n-2) \nabla \lambda^2 \exp(-2\lambda) \hat{g}_{ij} + (n-2) \lambda_i \lambda_j
\]

The existence of solutions to PDE system \( \hat{\mathbf{R}} = T \) where \( T \) is a given symmetric tensor is analyzed in [8]. It is instructive to consider this system from the point of view of overdetermined systems.

At the outset there are thus \( \frac{1}{4} n(n+1) \) second order quasilinear PDE for the components of the metric so that it seems the system is determined. However, we have the Bianchi identity (2.3) which should be taken into account. Let us then define the Bianchi operator \( B \) by the formula

\[
B(T) = 2 \text{div}(T) - \nabla \text{tr}(T) = 2g^{ij} T_{k;ij} - g^{ij} T_{ij;k}
\]
Hence if the system $R_i = T$ has solutions then necessarily the condition $B(T) = 0$ must also be satisfied. Note that this is a system of $n$ first order equations in metric $g$. Let us then define the modified Bianchi operator

$$\tilde{B}(T) = 2g^{ij}T_{ik,j} - g^{ij}T_{ij,k}$$

Note that the standard and covariant derivatives agree up to lower order corrections. Hence the components of $\tilde{B}(R_i)$ are second order differential operators. This leads to the following system:

$$\begin{align*}
R_i &= T \\
\tilde{B}(R_i) &= \tilde{B}(T) \\
B(T) &= 0
\end{align*}$$

Interestingly the initial system $R_i = T$ is not elliptic while the above completed system is, provided that $R_i$ is of full (matrix) rank. The ellipticity can then be used to show the existence of local solutions. In our case the system is

$$\begin{align*}
R_i &= (n-2)\lambda_{ij} \\
\tilde{B}(R_i) &= (n-2)\tilde{B}(\lambda_{ij}) \\
B(\lambda_{ij}) &= 0
\end{align*}$$

However, from the point of view of existence of solutions this is a very different system since $\lambda$ is also unknown, and moreover this is a third order system.

### 4 Setting up the computational problem

We will choose some families of metrics and try to construct metrics which satisfy one or more of the conditions in Definition 3.1. All conditions lead to systems of PDE whose nonlinearities are polynomial, and hence we can use `rifsimp` to analyse them.

It is known that the complexity of computing Gröbner basis is very bad (doubly exponential) in the worst case. Of course computing the involutive form is even more difficult. However, typically the time required for these computations is far from the worst case. On the other hand since there is no reasonable probability measure in the "space of all problems" there are no rigorous results on "average" complexity. Anyway all the solutions given below were obtained usually in few seconds and in any case in less than a minute with standard PC. Note that the decomposition of the system may take a lot more time than computing just the "most general" solution.

In all cases there actually were several components in the system, so that in the language of differential algebra the initial system was never prime differential ideal. It is not quite clear how to interpret this geometrically. Of course as components they provide essentially different solutions to PDE systems. From this it does not necessarily follow that the corresponding Riemannian manifolds are essentially different (i.e. not isometric). We did not attempt to study this problem. Since there were so many different components in all we will mostly give below only the most general one.

Let us now formulate more precisely the PDE systems that we are trying to solve.

**Problem 4.1 (RR problem)** Find a metric $g$ such that $P_{ijk} = 0$ where

$$P_{ijk} = sc R_{ij,k} - sc, k R_{ij}$$

This is a third order quasilinear system of $\frac{1}{2} n^2(n+1)$ PDE.

**Problem 4.2 (PRS problem)** Find a metric $g$ such that $Q_{ijk} = 0$ where

$$Q_{ijk} = 2sc R_{ij,k} - 2sc, k R_{ij} - sc, j R_{ik} - sc, j R_{ik}$$

This is a third order quasilinear system of $\frac{1}{2} n^2(n+1)$ PDE.

**Problem 4.3 (CO problem)** Find a metric $g$ such that the Cotton tensor $C = 0$. This is a third order quasilinear system of $n^2(n-1)$ PDE.

**Problem 4.4 (First QE problem)** Find a metric $g$ and a function $a$ such that the matrix rank of $T = R_i - a g$ is one. According to (2.4) there are $\binom{n^2}{2}$ algebraically independent fully nonlinear second order equations.
Problem 4.5 (Second QE problem) Find a metric $g$ and a function $\lambda$ such that $R_i = (n - 2)\nabla \nabla \lambda$. Here we have $\frac{1}{2} n(n+1)$ quasilinear second order equations in $\frac{1}{2} n(n+1) + 1$ unknowns. Note that in the completed system (3.6) there are integrability conditions which are expressed in terms of (modified) Bianchi operator. However, there is no need to compute them explicitly since rifsimp takes care of them automatically.

Note that the numbers of equations given above is the maximum number of algebraically independent PDE in the initial system. This number is achieved, if the metric is assumed to be completely general. However, if we assume that the metric is of specific form we may initially have less equations.

But the number of equations is in fact not really important in the present context. We recall that also with polynomial ideals the number of generators of the ideal does not matter, and even the number of generators of the Gröbner basis does not give any useful information. In the same way it is very convenient when using polynomial ideals the number of generators of the ideal does not matter, and even the number of generators in fact is not really important in the present context. We recall that also with rifsimp that it is not necessary to check beforehand if the equations are actually algebraically independent; rifsimp takes care of that automatically. Hence in practice we simply compute the relevant tensor and then require that all of its components are zero; if there are some redundant equations in the system then rifsimp simply discards them.

In the following we will look for the solutions of these systems. Since our analysis is local we will always consider only the situation in a single coordinate system. All our examples are of the form "separation of variables"; in other words all unknown functions are of functions of one variable only. In this way our PDE systems reduce to ODE systems, and it is thus easier to find solutions. The actual form of the initial guess of the metrics is not very critical. Experimenting with different choices showed that it is not particularly hard to find nontrivial solutions. Hence the reader can easily modify our examples and find other solutions.

In the actual computations the use of inequations was quite convenient. We may just look for those solutions where some function or a more complicated expression is non-zero. This is very natural in the problems below if for example we are not interested in cases where some unknown functions vanishes, or that the differential of the scalar curvature vanishes, or that the Ricci operator is zero. This can be speed up significantly the computations because then rifsimp does not need to worry about irrelevant subcases.

Let us now briefly describe the output of rifsimp. The algorithm tries to express highest ranking derivatives in terms of lower ranking derivatives. Let $f = (f_1, \ldots, f_k)$ be the unknown functions with $x = (x^1, \ldots, x^n)$ as independent variables. Let $\alpha^j$ be some multiindices. Then the first part of output is as follows:

$$\partial^{\alpha^j} f_j = F_j(x, f, \ldots) \quad 1 \leq j \leq m$$

In the arguments of $F_j$ there are only derivatives of lower ranking than $\partial^{\alpha^j} f_j$. The number $m$ is not known a priori. rifsimp also tries to eliminate the components $f_j$ from equations as far as possible. In linear algebra the Gaussian elimination reduces the problem to upper triangular form. Of course "differential nonlinear upper triangular form" does not exist in general, but rifsimp tries to compute a representation which is as close to it as possible. In the examples below it is seen clearly how this works. Note that the representation may depend heavily on the ranking chosen: the number $m$, multiindices $\alpha^j$ and functions $F_j$ are not intrinsic.

When the problems are nonlinear all the relevant information about the system cannot always be expressed as in (4.3). In these cases there are additional equations, called constraints in rifsimp, of the form

$$H_i(x, f, \ldots) = 0 \quad 1 \leq i \leq s$$

where the highest ranking derivatives of the arguments of $H_i$ are present non-linearly. Below we will see an example of this case also.

In addition to this there may be certain inequations in the output. When computing $F_j$ and $H_i$ sometimes one has to make some decisions if certain expressions are zero or not. This is how the system decomposes: in the generic case one assumes that "typically" any expression is nonzero. The algorithm keeps track of these assumptions and gives them in the output. But of course assuming that some expression is zero can produce solutions which are not contained in the "general" solution. Potentially there can be a lot of these branch points so that computing the decomposition, called casesplit in rifsimp, can take much more time than computing the generic solution.

Note that the output of rifsimp has a lot of structure and contains a lot of information. So even if one is unable to actually explicitly solve the equations given in the output one typically can immediately obtain some important facts which characterize the possible solution set. In many cases considered below the output can even be easily used for numerical computations while it is not at all clear how a numerical solution could be computed using the initial system.
Now the fact that \texttt{rifsimp} tries to approach the “upper triangular form” makes also the explicit solution easier. One can first solve equations with fewer variables, and then substitute these solutions to equations which contain more variables, like back substitution in Gaussian elimination. In solving the equations we often used the command \texttt{dsolve} in Maple.

5 Results

5.1 3 dimensional case

Let us consider the following metric:

\begin{equation}
\gamma = g_1(x^1)dx^1 + f_2(x^1)h_2(x^2)dx^2 + f_3(x^1)h_3(x^2)q(x^3)dx^3.
\end{equation}

\textbf{Example 5.1} Problem 4.1 with (5.1).

In this case our PDE system has a priori 18 independent equations but actually we have only 14 (not necessarily independent) nonzero equations. The system splits into seven subsystems. However, six systems either require that some unknown functions are constants, or the corresponding solutions give only the trivial solution where \( \beta \) reduces to zero. Note that it is anyway possible that in those cases there are solutions which are not special cases of the one given below.

The remaining component of the system has three differential equations; first two equations for \( f_j \):

\begin{align*}
f_2'' &= \frac{f_2(f_1^2 f_2 + f_1 f_3)}{2f_1 f_2} \\
 f_3'' &= \frac{f_3 f_2 f_3^2 + f_3 f_2 (f_3')^2 - f_1 f_3 f_2 f_3'}{2f_1 f_2 f_3}
\end{align*}

Evidently now one can give \( f_1 \) arbitrarily and then solve the remaining functions. However, one can actually eliminate one of the functions by solving \( f_1 \) and \( f_3 \) in terms of \( f_2 \) which gives the following family of solutions:

\( f_1 = \frac{c_2 f_2^2}{f_3} \quad \text{and} \quad f_3 = c_1 f_2^m. \)

Note that \( m \) need not be an integer. Then we have the third differential equation which contain \( h_2 \) and \( f_2 \).

However, when we substitute the above formulas the functions \( f_j \) disappear and we are left with

\( h_2'' = \frac{(2m-1)h_2 h_2' f_2 + mc_2 h_3 h_2' h_2'' - m^2 h_2^2 h_3^2}{2mc_2 h_2 h_3}. \)

Solving this for \( h_2 \) yields

\( h_2 = \frac{c_2 h_2^{1/m} h_2'^2}{h_3^2 (c_2 c_3 - m^2 h_3^{1/m})} = \frac{c_2 m^2 (h')^2}{h (c_2 c_3 - m^2 h)} \)

where we have introduced a new function \( h_3 = h^m \). Then writing \( f \) instead of \( f_2 \) we can write our final metric as

\( g = \frac{c_2 f^2}{f} (dx^1)^2 + \frac{m^2 c_2 f (h')^2}{h (c_2 c_3 - m^2 h)} (dx^2)^2 + c_1 f m^m q(dx^3)^2 \)

Clearly one can choose constants and functions such that \( g \) is positive definite. For scalar curvature we get \( sc = (1 - m)c_3/(2mfh) \) and thus \( \beta = -\nabla \ln(fh) \). Note that \( m \neq 1 \) because otherwise also \( R \) \( = 0 \).

\textbf{Example 5.2} Problem 4.2 with (5.1).

Now we have our PDE system which is very similar to RR case, but of course the solutions are different, by Theorem 3.2. Again we have 14 PDE, and computing with \texttt{rifsimp} get three cases where \( \alpha \neq 0 \). One case is the following:

\begin{align*}
f_2'' &= \frac{f_2 f_2'}{f_3} \\
 f_3'' &= \frac{f_3 (f_1 f_2^2 + f_3 f_2 f_3')}{2f_1 f_2 f_3} \\
 h_2'' &= \frac{h_2 h_2' h_3' + 3c_3 h_2'' + h_2'' h_3' h_2'' - h_2^2 (h_2')^2 - 2h_2 h_3 h_2' (h_2')^2 - 2h_2^2 (h_2')^2 h_2'}{2h_2^2 h_3}.
\end{align*}
It turns out that in the first two equations we can solve \( f_1 \) and \( f_2 \) in terms of \( f_3 \), and in the last one we get \( h_2 \) in terms of \( h_3 \):

\[
\begin{align*}
  f_1 & = \frac{c_1 (f_3')^2}{f_3} \\
  f_2 & = c_2 f_3 \\
  h_2 & = \frac{(h_3')^2}{h_3 (c_3 h_3 + c_4)}
\end{align*}
\]

Then writing \( f_3 = f \) and \( h_3 = h \) we get

\[
\begin{align*}
  g & = \frac{c_1 (f')^2}{f} (dx^1)^2 + \frac{c_2 f (h')^2}{h (c_3 h + c_4)} (dx^2)^2 + f h q (dx^3)^2 \\
  \alpha & = - \frac{f'}{2 f} \, dx^1 , \quad \text{Ri} = - \frac{c_1 c_3 + c_2 (c_2 (h')^2)}{4 c_1 c_2 h} \left( c_3 h + c_4 (dx^2)^2 + q h^2 (dx^3)^2 \right)
\end{align*}
\]

Note that the solution obtained satisfies also the QE condition with \( a \neq 0 \), and hence also the CO condition by Lemma 3.6.

**Example 5.3** Problem 4.3 with metric \((5.1)\).

Now we have 8 equations in the system. By computing with \texttt{rifsimp} get three cases and in the most general case we have

\[
\begin{align*}
  f_2'' & = \hat{F}(f_1, f_2, f_3) \\
  h_3'' & = H(f_1, f_2, f_3, h_2, h_3)
\end{align*}
\]

where \( \hat{F} \) and \( H \) are very complicated expressions, involving also the derivatives of its arguments, so that we do not write them down explicitly. The function \( H \) at the outset depends on \( f_j \) but if \( f_j \) satisfy the first equation then actually \( H \) does not depend on \( x^1 \). The dependence of \( H \) on \( f_j \) is only through initial conditions of the first equation, and consequently by standard theorems we have the local solution, and the above system can even be used for numerical computations.

However, it turns out that one can describe the solution in a more explicit way. One can actually solve the first equation for \( f_3 \) using quadratures which yields

\[
  f_3(x^1) = \exp \left( \hat{F}(f_1, f_2, f_1', f_2', f_3') \right)
\]

where in the expression \( \hat{F} \) there are also some integrals whose integrands depend on the variables indicated.

Now substituting this expression to the second equation gives

\[
2 h_2 h_3 h_3'' - h_2 h_3' h_3' - h_3'' (h_3')^2 + c_0 h_3^2 h_3'' = 0
\]

where \( c_0 \) is constant. Solving this gives

\[
  h_2 = \frac{(h_3')^2}{(c_1 - c_0 \ln(h_3)) h_3^2}
\]

Hence we can choose \( f_1 \), \( f_2 \), \( h_3 \) and \( g \) freely and it is clear that this choice, and the choice of constants \( c_0 \) and \( c_1 \), can be done in such a way that the metric is positive definite.

**Example 5.4** Problem 4.4 with metric \((5.1)\).

Constructing the appropriate PDE system we obtain five nonzero equations. Since the function \( a \) appears algebraically and in some equations even linearly we can solve it and substitute back to the equations. Note that one gets different families of solutions, depending on the choice of \( a \). However, we will analyze only one particular family of solutions.

After choosing \( a \) we are thus left with one single PDE; \texttt{rifsimp} gives then us the following system:

\[
\begin{align*}
  f_2'' & = F_1(f_1, f_2, f_3) \\
  f_3'' & = F_2(f_1, f_2, f_3) \\
  h_3'' & = H(f_1, f_2, f_3, h_2, h_3)
\end{align*}
\]
The expressions for $F_j$ and $H$ are again so big that we do not give them explicitly. Also the dependence of $H$ on $f_j$ is only through initial conditions as in the previous example, so that choosing $f_1$ and $f_2$ arbitrarily yields an ODE system in the standard form.

However, we can also solve the system explicitly; denoting $f_2 = f$ the first two equations give

$$f_1 = \frac{c_1 c_2 (f')^2}{f_3}$$

$$f_3 = c_3 m^{-m} f^{1-m} (c_2 f - 1)^m$$

Substituting this into third equations yields

$$h''_3 = \frac{((3m - 2)h_2 h'_3 + 2(m - 1)h_3 h'_2) h'_3}{4(m - 1)h_2 h_3}$$

Denoting $h_3 = h$ and solving for $h_2$ yields

$$h_2 = c_4 h^{(2-3m)/(2m-2)} (h')^2$$

After this it is straightforward to compute $a$, $b$ and $\omega$ which gives

$$a = \frac{m}{8(m - 1)c_4} h^{(2-m)/(2m-2)} - \frac{c_2^2 f^2 + (m - 2)c_2 f + (m - 1)^2}{2c_1 m^m f^m} (c_2 f - 1)^{m-2}$$

$$b = \frac{m}{8(m - 1)c_4} h^{(2-m)/(2m-2)} + \frac{m(m - 1)}{2c_1 m^m f^m} (c_2 f - 1)^{m-2}$$

$$\omega = f (c_2 f - 1) \partial_{\alpha_1} + (2 - 2m) f f' \partial_{\alpha_2}$$

**Example 5.5** Problem 4.5 with metric (5.1).

Here we see that $\lambda_2$, $h_3$ and $f_2$ must be constants; for simplicity let us choose $\lambda_2 = h_3 = f_2 = 1$. Then for other functions we obtain

$$f_1 = c_1 (\lambda'_j)^2$$

$$f_3 = c_2 \lambda'_j$$

$$q_3 = c_1 (\lambda'_j)^2$$

Note that there is no condition on $h_2$, and also $\lambda_1$ and $\lambda_3$ can be freely chosen. This solution implies that $\mathbf{Ri} = \nabla \nabla \lambda = 0$, but of course $\mathbf{Ri}$ gives a nontrivial example of QE manifold.

**5.2 First 4 dimensional case**

Let us then consider a simple four dimensional case:

$$g = (dx^1)^2 + f(x^1) q(x^1)(dx^2)^2 + (dx^3)^2 + (dx^4)^2)$$

**Example 5.6** Problem 4.1 with metric (5.2).

It turns out that RR system has only solutions with $\mathbf{Ri} = 0$ so there are no examples of this form. Here the use of inequations was very convenient. When one added to the PDE system the condition $\beta \neq 0$, nfsimp concluded that the system is inconsistent.

**Example 5.7** Problem 4.2 with metric (5.2).

This illustrates quite well how nfsimp handles the system and how the solutions can split into several (in this case two) families, so that we describe this in more detail. The equations of the PRS system give

$$f'' = \frac{(f')^2}{2f}$$

$$q'' = \frac{4q q' - 3(q')^3}{q^2}$$

$$4f^2 (q'')^2 - 10fq(q')^2 q'' + 6f(q')^4 + 2q^2 (f')^2 q'' - 3q^2 (f')^2 (q')^2 = 0$$

(5.3)
The solution (5.4) satisfies also the QE condition and hence by Lemma 3.6 also the condition CO. On the other hand the Ricci tensor corresponding to the solution (5.5) has two double eigenvalues and hence cannot be QE.

**Example 5.8** Problem 4.3 with metric (5.2).

The CO case is very easy: \( f \) is arbitrary and

\[
q'' = \frac{3(q')^2}{2q} \quad \Rightarrow \quad q = \frac{1}{(c_1 x^4 + c_0)^{\frac{1}{2}}}
\]

Here we get the same \( q \) as in (5.4). Hence here PRS case is a subcase of CO case.

**Example 5.9** Problem 4.4 with metric (5.2).

After computing the minors we solve \( a \) from one of the equations which yields

\[
a = -\frac{2f f'' q^3 f + (f')^2 q^3 + 2fq q'' - (q')^2 f}{4 f q^3}
\]

Substituting this expression to the system and then computing with rifsimp gives the following system for \( f \) and \( q \):

\[
f''' = -\frac{2f f'' f' + (f')^3}{f^2}
\]

\[
q'' = \frac{4f'' q^3 f - 4(f')^2 q^3 + (q')^2 f}{2q f}
\]

The first equation can be solved and then the second one is in the standard form:

\[
f = 4c_3 \cosh(c_1 x^4 + c_2)^2
\]

\[
q'' = \frac{32c_3^2 c_3 q^3 + (q')^2}{2q}
\]

Interestingly if \( 4c_3^2 c_3 = 1 \) then \( q \) is a Weierstrass elliptic function.
Let us now consider 5.3 Second 4 dimensional case

Here it is natural to suppose that \( \lambda = \lambda_1(x^4)\lambda_4(x^4) \). The most general solution is now incompatible with the positive definiteness of the metric. However, we still have a nontrivial family solutions. First we set \( \lambda_4 = 1 \) and then compute \( q = 1/(c_1x^4 + c_0)^2 \). So here again \( q \) must be the same as in CO case and in one of the PRS cases. Hence all metrics obtained in this way must satisfy also the CO condition.

Substituting the computed value of \( q \) to the system leaves us with the following equations:

\[
2ff'' + ff'\lambda_1' + 8c_1^2f + (f')^2 = 0
\]
\[
4f^2\lambda_1'' - 6ff'\lambda_1' - 12c_1^2f - 3(f')^2 = 0
\]

There is no explicit formula for solution but again by standard theorems the local solution exists. Note that the metrics obtained in this way satisfy the QE condition. Hence we have metrics \( g \) and \( \hat{g} \) which are both QE and which are conformally equivalent.

5.3 Second 4 dimensional case

Let us now consider

\[
g = f_1(x^4)(dx^1)^2 + f_2(x^4)(dx^2)^2 + f_3(x^4)(dx^3)^2 + f_4(x^4)(dx^4)^2
\]

Example 5.11 Problem 4.1 with metric (5.6).

It turns out that RR conditions force two of the functions \( f_2, f_3 \) and \( f_4 \) to be constants. Choosing for example \( f_3 = f_4 = 1 \) we have \( \beta = \nabla \ln(sc) \) where

\[
sc = \frac{-2f_1f_2f_3' + f_1(f_2')^2 + f_2f_3f_3'}{2f_1f_2f_3}
\]

In essence the problem reduces to the 2 dimensional case and of course in 2 dimensions any metric is RR.

Example 5.12 Problem 4.2 with metric (5.6).

The system decomposes to 8 components and the most general one gives the following system:

\[
f_2'' = \frac{(f_2f_3f_4f_1' + f_1f_2f_4f_2' - f_1f_2f_3f_3' - f_1f_2f_3f_3')f_3'}{2f_1f_2f_3f_4}
\]
\[
f_3'' = \frac{(f_2f_3f_4f_1' - f_1f_2f_4f_2' + f_1f_2f_3f_3' - f_1f_2f_3f_3')f_3'}{2f_1f_2f_3f_4}
\]
\[
f_4'' = \frac{f_2f_3f_4f_1' + 2f_1(f_2')^2f_3' + f_1f_2f_3f_3' + f_1f_2f_3f_3' + f_1f_2f_3f_3'}{2f_1f_2f_3f_4}
\]

It turns out that one can solve this explicitly. Let us set \( f_3 = f \); then the other functions are given by

\[
f_1 = c_1c_4m^2 \exp((c_3f^{m+1})f^{(m^2 - m - 1)/(m+1)}(f')^2
\]
\[
f_2 = c_2f^m
\]
\[
f_4 = c_4 \exp((c_3f^{m+1})f^{-m/(m+1)}
\]

This yields

\[
\alpha = \frac{(m - c_3(m+1)^2f^{m+1})f'}{2(m+1)f} dx^1
\]
\[
\beta = \frac{c_3(m+1)^2}{2c_1c_2m^2} (dx^4)^2
\]

Note that \( \beta \) also satisfies the QE condition with \( \alpha = 0 \), see Theorem 3.5.

Example 5.13 Problem 4.3 with metric (5.6).

In the CO case we have initially 6 equations, but \texttt{rifsimp} gives only the following 2 equations:

\[
f_2'' = F_2(f_1, f_2, f_3, f_4)
\]
\[
f_3'' = F_3(f_1, f_2, f_3, f_4)
\]

Here the expressions of \( F_2 \) and \( F_3 \) are so big that we do not give them explicitly. Anyway choosing \( f_1 \) and \( f_4 \) arbitrarily we have a standard ODE system for \( f_2 \) and \( f_3 \).
Example 5.14 Problem 4.4 with metric (5.6).

There are three families of solutions and as usual we give the most general. First rifsimp gives:

\[
\begin{align*}
\rho'' & = \frac{2 f_1 f_2 f_3 f_4 f'' - f_1 f_2 f_3 (f_1')^2 + f_1 f_2 f_4 f'_4 + f_1 f_3 f_4 f'_2 - f_2 f_3 f_4 f'_2}{2 f_1 f_2 f_3 f_4} \\
\sigma'' & = \frac{2 f_1 f_2 f_3 f_4 f'' - f_1 f_2 f_3 (f_1')^2 + f_1 f_2 f_4 f'_4 + f_1 f_3 f_4 f'_2 - f_2 f_3 f_4 f'_2}{2 f_1 f_2 f_3 f_4}
\end{align*}
\]

If \( f_1 \) and \( f_4 \) are given arbitrarily this is in standard form so the local solution exists. Then we see that we can choose \( \omega = dx^1 \), and \( a \) and \( b \) are given by

\[
\begin{align*}
a &= \frac{f_1 f_2 f_3 (f_1')^2 + f_2 f_3 f_4 f'_4 + f_1 f_2 f_4 f'_2 - f_1 f_3 f_4 f'_2 - 2 f_1 f_2 f_3 f_4 f''}{4 f_1 f_2 f_3 f_4} \\
b &= \frac{f_1 f_2 f_3 (f_1')^2 + f_2 f_3 f_4 f'_4 + f_1 f_2 f_4 f'_2 - 2 f_1 f_2 f_3 f_4 f''}{2 f_1 f_2 f_3 f_4}
\end{align*}
\]

Example 5.15 Problem 4.5 with metric (5.6).

Here we choose that \( \lambda \) also is only function of \( x^1 \). This gives the system

\[
\begin{align*}
\rho'' &= \frac{f_1 f_2 f_3 (f_1')^2 + f_2 f_3 f_4 f'_4 + f_1 f_2 f_4 f'_2 - f_1 f_3 f_4 f'_2 - 4 f_1 f_2 f_3 f_4 f''}{2 f_1 f_2 f_3 f_4} \\
\sigma'' &= \frac{f_1 f_2 f_3 (f_1')^2 + f_2 f_3 f_4 f'_4 + f_1 f_2 f_4 f'_2 - f_1 f_3 f_4 f'_2 - 4 f_1 f_2 f_3 f_4 f''}{2 f_1 f_2 f_3 f_4}
\end{align*}
\]

This looks complicated but actually we can solve it in terms of \( f_2 \). So denoting \( f = f_2 \) and \( \nu = \sqrt{1 + n^2 + m^2} \) we obtain

\[
\begin{align*}
f &= c_3 f_4 \\
\lambda &= \frac{1 + n + m + \nu}{4} \ln(f)
\end{align*}
\]

### 5.4 Third 4 dimensional case

Let us consider the following four dimensional case:

\[
g = q(x^1)(dx^1)^2 + u(x^3)(dx^3)^2 + h(x^2)(dx^3)^2 + f(x^1)(dx^1)^2
\]

Example 5.16 Problem 4.1 with metric (5.7).

In the RR case we have 16 equations. Again there are sub cases but the most general solution is the following. For \( h \) and \( u \) we have the equations

\[
\begin{align*}
h'' &= \frac{h'(2 h'' - (h')^2)}{2 h} \\
u'' &= \frac{u(h')^2 + h(u')^2 - 2 h u h''}{2 h u}
\end{align*}
\]

Again the second equation depends on \( h \) only through initial conditions. Hence by standard theorems local solutions exist for these equations. Then \( \beta \) is of the form \( \beta = \beta_1 x^1 + \beta_4 x^4 \) where \( \beta_j \) depend only on \( f \) and \( q \).
Example 5.17 Problem 4.2 with metric (5.7).
The system is incompatible with the requirement \( \alpha \neq 0 \), so there are no PRS metrics of this form.

Example 5.18 Problem 4.3 with metric (5.7).
In this case the equations yield only solutions where \( \nabla R_i = 0 \), so there are no interesting examples of this form.

Example 5.19 Problem 4.4 with metric (5.7).
Computing the minors and solving for \( a \) we obtain a system which requires that two of the functions \( f, h, u \), and \( q \) must be constants. This leads to a solution with nontrivial Ricci tensor, but this Ricci tensor is necessarily Einstein so there are no nontrivial quasi Einstein metrics of this form.

Example 5.20 Problem 4.5 with metric (5.7).
Here the most general solution to the differential equations is incompatible with positive definiteness of the metric. For the next general system one has \( \lambda_4 = \lambda_3 = f = 1 \) and for other functions we obtain

\[
\begin{align*}
2h^2h'' - h' \left( 2hh'' - (h')^2 \right) &= 0 \\
2qq'' - (q')^2 &= 0 \\
2huh'' + 2hkh'' - h(u')^2 - u(h')^2 &= 0 \\
4q\lambda_4'' + (q')^2\lambda_1 &= 0 \\
2q\lambda_4' - \lambda_4 q' &= 0
\end{align*}
\]

This has the following family of solutions

\[
\begin{align*}
q &= (c_1 x^4 + c_0)^2 \\
u &= (c_2 x^3 + c_3)^2 \\
h &= (c_4 x^2 + c_5)^2 \\
\lambda_4 &= c_6 (c_1 x^4 + c_0) \\
\lambda_1 &= c_7 \cos(c_1 x^4) + c_8 \sin(c_1 x^4)
\end{align*}
\]

Here \( R_i = \nabla \nabla \lambda = 0 \), but of course \( \hat{R}_i \) is nontrivial.

6 Conclusion

We have seen above that with appropriate methods one can readily get large classes of nontrivial examples of various classes of Riemannian manifolds. The particular initial form of the Riemannian metric was not critical; testing different choices revealed that typically one could always find solutions. The formal theory of PDE can be also useful for exploring other classes of Riemannian manifolds, and in general related questions in differential geometry. Spivak writes in [21, p. 189] that many problems in differential geometry are in fact problems of overdetermined PDE whose solution require that one knows all the integrability conditions. Since manipulating PDE systems by hand is typically extremely tedious, one has invented "incredibly concise and elegant ways to state the integrability conditions [...] without ever even mentioning partial derivatives." However, the tools which were used above to compute the involutive form of various systems can perhaps be helpful in many other contexts in differential geometry, at least in producing relevant (counter)examples in various situations.

References

MARYAM SAMAVAKI

This thesis presents some new solutions based on linearized the homogeneous Navier-Stokes equations on Riemannian manifolds, related to the Lie bracket of Killing vector fields. The equations for the Killing and conformal Killing vector fields which are overdetermined systems of PDE can be formulated as an eigenvalue problem. Moreover, analyzing several classes of Riemannian manifolds can be interpreted as overdetermined PDE systems whose unknowns are the Riemannian metric components.