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A Dynamic Theory of a Consumer

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Abstract

The static neoclassical theory of a consumer and its dynamization by dynamic optimization yield equal results. On the other hand, the Ramsey (1928) macro model of consumption dynamics does not explain the dynamics of real consumption of an individual consumer. As a solution to these problems, we present a dynamic theory of a consumer consistent with the static neoclassical theory. We define the 'economic force' by which the consumer acts upon his consumption and show that the adjustment in a utility-seeking way may be stable or unstable. The proposed framework allows the modelling of economic growth together with optimal behavior as is assumed in the static neoclassical framework. (JEL D21, O12)

Keywords: Consumption dynamics, economic force, instability.

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1 Introduction

Mirowski (1989b) shows that the progenitors of neoclassical economics consciously imitated classical mechanics. The concept of equilibrium, for example, was introduced in economics from physics by Canard in 1801 (Mirowski 1989a). Although equilibrium means a 'balance of forces' situation, in economics the balancing 'forces' have not been defined. In spite of this, the use of the term 'force' is common in economics; see Lucas (1988) for example. The 'invisible hand' by Adam Smith, for instance, serves as an example of how the concept of 'force field' has been applied in economics.

Static neoclassical theory as a whole is an application of equilibrium analysis. Static analysis is not, however, in accordance with the observed evolutionary behavior of economies. The assumption that economic agents behave in an optimal way prohibits the understanding of dynamics because no agent likes to change his optimal behavior. In physics, too, the equilibrium states of various dynamic systems were understood before Newton defined his dynamic laws where equilibrium states are special cases of dynamic systems.

Here we propose a new framework for modelling in economics by assuming that *economic agents like to better their situation when possible.* We believe that the willingness of economic agents to better their situation is the fundamental cause of economic dynamics. We demonstrate the applicability of this framework in modelling consumer behavior.

Our approach offers three advantages compared with the existing principles of economic modelling: 1) Dynamic optimization as a mathematical technique is not needed. 2) Static neoclassical theory is a special case in our modelling — the zero-force situation — and so only one framework is needed. 3) Our framework covers also cases where a static optimum does not exist.

2 Static Neoclassical Theory of a Consumer

We assume a consumer's decision-making situation as simple as possible. The length of the time horizon of the consumer is assumed to be one week, and the consumer can choose his weekly consumption of only two goods the consumer consumes every week. For clarity, let good 1 be 'food' and good 2 'playing video games' according to the traditional choice between 'food or fun'. The consumer is assumed to have budgeted a fixed amount of money T ($\$/week$) for his weekly consumption, and the unit prices of food and playing video games are p_1 ($\$/kg$) and p_2 ($\$/h$), respectively[1]. The weekly budget

[1] A system of measurement units for economics is given in de Jong (1967). Measurement units are in braces after the quantities.

of the consumer is then $T = p_1 q_1 + p_2 q_2$ where the consumption flows of the two goods are denoted by q_1 ($kg/week$) and q_2 ($h/week$).

The consumer has a continuous scalar valued weekly utility function $u = f(q_1, q_2)$. To be able to write well-defined mathematical expressions with the utility function, we give measurement unit ut for utility. The consumer spends all the money he has budgeted for his consumption for the week during the week, and so the satisfaction he gains from his consumption takes place at the week. The values of utility function $u = f(q_1, q_2)$ are then measured in units $ut/week$. Utility u thus measures the average flow of satisfaction the consumer gains during the week.

The explicit measuring of utility is not needed in modelling consumer behavior, however. Utility is only an auxiliary quantity required in defining the willingness-to-pay of a consumer for various things. Every utility function that expresses the same preference order of a consumer defines a measurement unit for utility according to the values of the function. However, all utility functions expressing the same preference order of a consumer give the same 'willingness-to-pay' values for goods near the consumer's optimum, see the Appendix. The actual measuring problems of the level of satisfaction of a consumer can thus be omitted with these remarks.

The *average utility of a good for a consumer* is measured by dividing the utility of the consumer by his consumption of the good at the time unit. The average utility of food and playing video games in a week are thus

$$\frac{u}{q_1} = \frac{f(q_1, q_2)}{q_1} \quad \text{and} \quad \frac{u}{q_2} = \frac{f(q_1, q_2)}{q_2}$$

with units $(ut/week)/(kg/week) = ut/kg$ and $(ut/week)/(h/week) = ut/h$, respectively; they measure the average satisfaction the consumer gains from one kilogram of food and one hour of playing video games at the week.

The consumer's marginal utilities of the two goods are

$$\frac{\partial f(q_{1_0}, q_{2_0})}{\partial q_1} > 0, \quad \frac{\partial f(q_{1_0}, q_{2_0})}{\partial q_2} > 0,$$

where q_{1_0}, q_{2_0} are fixed flows of consumption. The measurement unit of the marginal utility of a good is the same as that of average utility.

From the weekly budget equation we get $q_2 = \big(T - p_1 q_1\big)/p_2$. Substituting this in the utility function gives

$$u = f\big(q_1, q_2\big) = f\left(q_1, \frac{T - p_1 q_1}{p_2}\right) \equiv F(q_1, T, p_1, p_2). \tag{2}$$

Static neoclassical theory assumes optimal behavior. The optimal weekly consumption of food q_1^* can be solved from the following equation

$$\frac{du}{dq_1} = 0 \quad \Leftrightarrow \quad \frac{\partial f}{\partial q_1} - \frac{p_1}{p_2}\frac{\partial f}{\partial q_2} = 0 \quad \Leftrightarrow \quad \frac{1}{p_1}\frac{\partial f}{\partial q_1} = \frac{1}{p_2}\frac{\partial f}{\partial q_2}, \tag{3}$$

which can also be presented according to (2) as

$$\frac{\partial F}{\partial q_1} = \frac{\partial f}{\partial q_1} - \frac{p_1}{p_2}\frac{\partial f}{\partial q_2} = 0.$$

A sufficient condition for maximum is that

$$\frac{d^2u}{dq_1^2} = \frac{\partial^2 f}{\partial q_1^2} - \frac{\partial^2 f}{\partial q_2 \partial q_1}\frac{p_1}{p_2} + \frac{\partial^2 f}{\partial q_2^2}\left(\frac{p_1}{p_2}\right)^2 - \frac{\partial^2 f}{\partial q_1 \partial q_2}\frac{p_1}{p_2} < 0. \tag{4}$$

Non-increasing marginal utility makes $\partial^2 f/\partial q_1^2$, $\partial^2 f/\partial q_2^2$ non-positive. The first and third additive terms in (4) are thus non-positive. If the partial functions of a multi-variable function are continuous, then $\partial^2 f/(\partial q_1 \partial q_2) = \partial^2 f/(\partial q_2 \partial q_1)$ holds (Apostol (1979) p. 360). Now, assuming the partial functions of the utility function continuous, the sufficient condition for maximum is that $\partial^2 f/\partial q_2 \partial q_1 > 0$. Thus the greater the flow of food consumption, the more the consumer enjoys increasing his playing of video games when he consumes in the limits of his budget.

3 Dynamic Theories of Consumption

3.1 Dynamization by Dynamic Optimization

A common principle to model economic dynamics is to assume that economic agents maximize their target functions over a finite or infinite future. We continue analyzing the two-good situation and now we assume that the consumption flows of the two goods depend on time t with unit *week*. From the weekly budget equation we get $q_2(t) = \big(T - p_1 q_1(t)\big)/p_2$, where the other quantities except the consumption flows are assumed fixed. Substituting this in the utility function gives

$$u(t) = f\left(q_1(t), \frac{T - p_1 q_1(t)}{p_2}\right) \equiv F(q_1(t), T, p_1, p_2). \tag{5}$$

Assuming that the consumer lives an infinite time, the dynamic optimization problem of the consumer becomes the following

$$\max_{q_1(t)} \int_0^\infty u(t)dt = \max_{q_1(t)} \int_0^\infty F(q_1(t), T, p_1, p_2)dt.$$

The Euler equation of this dynamic optimization problem is

$$\frac{\partial F}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial F}{\partial q_1'(t)}\right) = 0 \quad \Rightarrow \quad \frac{\partial F}{\partial q_1} = 0.$$

The necessary condition for this optimization problem equals with Eq. (3) because quantity $q_f'(t)$ does not exist in function F. Dynamic optimization as a technique thus does not give an equation of motion for the consumer's food consumption as was expected. For dynamic optimization to give an equation of motion for the consumption of a consumer, either the target function or the budget equation must be changed from that of static analysis. However, then the two frameworks of modelling are not consistent with each other.

3.2 The Ramsey Model

The inconsistency described in the previous section has lead to a situation that a dynamic model for the real consumption of a consumer does not exist. On the other hand, the dynamics of consumption has been modelled at the aggregate level by the model of Ramsey (1928). The Ramsey model is presented here according to Chiang (1992 pp. 111-116) because of the more convenient notation. Ramsey assumed that an economy can either save or consume all production,

$$C(t) = Q\big(K(t), L(t)\big) - K'(t), \ S(t) = K'(t) \ \Rightarrow \ C(t) + S(t) = Q\big(K(t), L(t)\big)$$

where C is aggregate consumption, K aggregate capital, Q aggregate production, L aggregate amount of labor available and S aggregate saving of the economy. Social utility is measured by function $U(C)$ with $U'(C) > 0$, $U''(C) < 0$. In the production of consumption goods, the society needs labor which causes disutility $D(L)$. Net social utility $N(C, L)$ is then

$$N(C, L) = U(C) - D(L).$$

The economic planner's problem is to minimize the deviation of the social utility for the current and all future generations to come from the maximum possible attainable utility denoted by B (Bliss):

$$\min_{L(t),K(t)} \int_0^\infty [B - U(C(t)) + D(L(t))]dt, \quad C(t) = Q\big(K(t), L(t)\big) - K'(t)$$

subject to $K(0) = K_0$. The integrand is thus

$$G(t) = B - U\Big(Q\big(K(t), L(t)\big) - K'(t)\Big) + D(L(t)).$$

The Euler equations for this dynamic minimization problem are:

$$\frac{\partial G}{\partial L} - \frac{d}{dt}\left(\frac{\partial G}{\partial L'(t)}\right) = 0 \quad \Leftrightarrow \quad -U'(C)\frac{\partial C}{\partial L} + D'(L) = 0, \tag{6}$$

$$\frac{\partial G}{\partial K} - \frac{d}{dt}\left(\frac{\partial G}{\partial K'(t)}\right) = 0 \quad \Leftrightarrow \quad -U'(C)\frac{\partial C}{\partial K} - \frac{d}{dt}U'(C) = 0, \tag{7}$$

where $\partial G/\partial L'(t) = 0$, $\partial C/\partial L = \partial Q/\partial L$ and $\partial C/\partial K = \partial Q/\partial K$. Writing Eq, (6) as

$$U'(C)\frac{\partial Q}{\partial L} = D'(L)$$

we see that in the optimum the marginal disutility from work equals the product of marginal utility of consumption and marginal productivity of labor. Eq. (7) can be rewritten as

$$-\frac{\frac{d}{dt}\left(U'(C)\right)}{U'(C)} = \frac{\partial Q}{\partial K}. \tag{8}$$

We can read this so that the growth rate of the marginal utility of consumption must at every point in time equal with the marginal productivity of capital. Another way to understand Eq. (8) is to write the left hand side as

$$-\frac{\frac{d}{dt}\left(U'(C)\right)}{U'(C)} = -\frac{U''(C)}{U'(C)}C'(t)$$

where $U''(C) < 0$. Then Eq. (8) becomes

$$C'(t) = -\frac{\partial Q}{\partial K}\frac{U'(C)}{U''(C)}. \tag{9}$$

From (9) we can read that consumption increases $C'(t) > 0$ the faster the greater the marginal productivity of capital $\partial Q/\partial K$ and the higher the marginal utility of consumption $U'(C)$.

The Ramsey model explains the dynamics of the aggregate money flow directed for consumption in an economy, and so it does not dynamize the static neoclassical theory of the real consumption of a consumer.

4 A Dynamic Theory of Real Consumption

Here we model dynamic consumer behavior so that the optimal behavior assumed in the static neoclassical framework corresponds to an equilibrium state in this. We continue analyzing the two-good situation, and let the consumer's weekly utility function be as in section 3.1, $u(t) = f\big(q_1(t), q_2(t)\big) = f\left(q_1(t), \frac{T-p_1q_1(t)}{p_2}\right)$. The consumer is assumed to adjust his consumption flows of the two goods to increase his weekly utility with time. The consumer can now affect his weekly utility only by quantity q_1 because other quantities in the function are constants and q_2 is substituted by the budget equation. Differentiating the function with respect to time, we get

$$u'(t) = \frac{du}{dq_1}q_1'(t) = \left(\frac{\partial f}{\partial q_1} - \frac{p_1}{p_2}\frac{\partial f}{\partial q_2}\right) q_1'(t). \tag{10}$$

In (10), the unit of $u'(t)$ is $ut/week^2$ and that of $q_1'(t)$ is $kg/week^2$; $u'(t)$ and $q_1'(t)$ are thus the instantaneous acceleration of utility and food consumption while u and q_1 are the corresponding velocities.

The consumer is assumed to adjust his consumption with time to increase his weekly utility. The adjustment rules for food consumption are: $q_1'(t) > 0$ if $\frac{du}{dq_1} > 0$, $q_1'(t) < 0$ if $\frac{du}{dq_1} < 0$ and $q_1'(t) = 0$ if $\frac{du}{dq_1} = 0$. These adjustments make the right hand side of Eq. (10) positive and then the weekly utility increases with time, $u'(t) > 0$. The condition for the equilibrium state $q_1'(t) = 0$,

$$\frac{du}{dq_1} = 0 \quad \Leftrightarrow \quad \frac{\partial f}{\partial q_1} - \frac{\partial f}{\partial q_2}\frac{p_1}{p_2} = 0 \quad \Leftrightarrow \quad \frac{\partial f}{\partial q_1}\frac{1}{p_1} = \frac{\partial f}{\partial q_2}\frac{1}{p_2}, \tag{11}$$

corresponds to the optimal state of the consumer given earlier.

Because $\frac{\partial f}{\partial q_1}$, $\frac{\partial f}{\partial q_2}$, p_1 and p_2 are positive, multiplying the consumer's adjustment inequalities of food consumption by factor $p_2/\frac{\partial f}{\partial q_1} > 0$ we get

$$\begin{aligned} q_1'(t) > 0 \quad &\text{if} \quad \frac{p_2}{\frac{\partial f}{\partial q_2}}\frac{\partial f}{\partial q_1} - p_1 > 0, \\ q_1'(t) < 0 \quad &\text{if} \quad \frac{p_2}{\frac{\partial f}{\partial q_2}}\frac{\partial f}{\partial q_1} - p_1 < 0, \\ q_1'(t) = 0 \quad &\text{if} \quad \frac{p_2}{\frac{\partial f}{\partial q_2}}\frac{\partial f}{\partial q_1} - p_1 = 0. \end{aligned}$$

Analogous adjustment rules can be derived for playing video games. This is done as follows. Solve the budget equation with respect to $q_1(t)$, use this to substitute $q_1(t)$ from the utility function and differentiate it with respect to time. After this, define the adjustment rules as we did above. Quantities

$$h_1 = \left(\frac{\partial f}{\partial q_1}/\frac{\partial f}{\partial q_2}\right) p_2, \quad h_2 = \left(\frac{\partial f}{\partial q_2}/\frac{\partial f}{\partial q_1}\right) p_1$$

derived in this way have units $\$/kg$ and $\$/h$, respectively, and we can interpret them as this *consumer's willingness-to-pay* for one kilogram of food and for one hour of playing video games, respectively. The explanation is following. A utility-seeking consumer compares the above quantities and the prices of the goods, and increases the consumption of that good for which the above quantity is greater than the price, and decreases the consumption of that good for which the quantity is smaller than the price. The consumer must pay the price of the good the consumption of which he increases, and he does not pay the price of the good the consumption of which he decreases. Thus when $h_1 > p_1$, a utility-seeking consumer pays the price of good 1. This behavior is empirically testable by making a questionnaire about consumers' willingness-to-pay for a good and comparing these with its price.

A consumer's willingness-to-pay for food is the greater the higher the $\partial f/\partial q_1$ and the smaller the quantity $\frac{\partial f}{\partial q_2}/p_2$ with unit $ut/\$$. The latter measures the consumer's *marginal utility of budgeted funds for the week*. We can show this by differentiating the utility function in (2) with respect to T,

$$\frac{\partial u}{\partial T} = \frac{\partial f}{\partial q_2}\frac{1}{p_2}.$$

If we substitute q_1 from the utility function by using the budget equation, we get for the marginal utility of budgeted funds as $\frac{\partial u}{\partial T} = \frac{\partial f}{\partial q_1}/p_1$. In the consumer's optimum these two quantities are equal, see (3).

In the Appendix we show that a consumer's willingness-to-pay for a good is independent on the chosen utility function: *any continuous function expressing the same preference order gives an equal willingness-to-pay for a good in the neighborhood of a consumer's optimum.* The ambiguity in measuring utility thus does not affect our modelling because different marginal utilities given by different utility functions divided by marginal utilities of budgeted funds by the same utility function give equal willingness-to-pay values.

4.1 A Consumer's Willingness-To-Pay and Demand

A consumer's willingness-to-pay for food

$$h_1 = \left(\frac{\partial f(q_1, q_2)}{\partial q_1} / \frac{\partial f(q_1, q_2)}{\partial q_2}\right) p_2 \tag{12}$$

has the following characteristics:

$$\frac{\partial h_1}{\partial q_1} = \frac{\left(\frac{\partial^2 f}{\partial q_1^2} - \frac{\partial^2 f}{\partial q_1 \partial q_2}\frac{p_1}{p_2}\right)\frac{\partial f}{\partial q_2} - \frac{\partial f}{\partial q_1}\left(\frac{\partial^2 f}{\partial q_2 \partial q_1} - \frac{\partial^2 f}{\partial q_2^2}\frac{p_1}{p_2}\right)}{\left(\frac{\partial f}{\partial q_2}\right)^2} p_2, \tag{13}$$

$$\frac{\partial h_1}{\partial T} = \frac{\frac{\partial^2 f}{\partial q_2 \partial q_1}\frac{\partial f}{\partial q_2} - \frac{\partial f}{\partial q_1}\frac{\partial^2 f}{\partial q_2^2}}{\left(\frac{\partial f}{\partial q_2}\right)^2}, \tag{14}$$

$$\frac{\partial h_1}{\partial p_2} = \frac{\left(\frac{\partial^2 f}{\partial q_2^2}\frac{\partial f}{\partial q_1} - \frac{\partial^2 f}{\partial q_2 \partial q_1}\frac{\partial f}{\partial q_2}\right)\left(\frac{T - p_1 q_1}{p_2}\right)}{\left(\frac{\partial f}{\partial q_2}\right)^2} + \frac{\frac{\partial f}{\partial q_1}}{\frac{\partial f}{\partial q_2}}. \tag{15}$$

The law of non-increasing marginal utility

$$\frac{\partial^2 f}{\partial q_1^2} \leq 0, \quad \frac{\partial^2 f}{\partial q_2^2} \leq 0,$$

and the positiveness of the second order cross partial

$$\frac{\partial\left(\frac{\partial f}{\partial q_2}\right)}{\partial q_1} = \frac{\partial^2 f}{\partial q_1 \partial q_2} = \frac{\partial^2 f}{\partial q_2 \partial q_1} = \frac{\partial\left(\frac{\partial f}{\partial q_1}\right)}{\partial q_2}$$

make $\partial h_1/\partial q_1 < 0$ and $\partial h_1/\partial T > 0$. In $\partial h_1/\partial q_1$, the first term in braces in the numerator is negative, the latter term in braces is positive and the denominator is positive. The condition for $\partial h_1/\partial q_1 < 0,\ \partial h_1/\partial T > 0$ is thus the same as that the equilibrium point maximizes the consumer's weekly utility. In $\partial h_1/\partial p_2$, the first additive term is negative and the latter is positive and so the sign is ambiguous.

The following equation corresponds to the consumer's optimum,

$$p_1 = h_1 \quad \Leftrightarrow \quad p_1 = \left(\frac{\partial f(q_1, q_2)}{\partial q_1} / \frac{\partial f(q_1, q_2)}{\partial q_2}\right) p_2. \tag{16}$$

We call Eq. (16) the *demand relation for food of this consumer*. The demand relation is similar to that of the willingness-to-pay, but their slopes in coordinate system $(q_1, \$/kg)$ differ. We show this next. By totally differentiating

Eq. (16) and using the utility function in (2), we get

$$\left(1+\frac{\left(\frac{\partial^2 f}{\partial q_2\partial q_1}\frac{\partial f}{\partial q_2}-\frac{\partial^2 f}{\partial q_2^2}\frac{\partial f}{\partial q_1}\right)q_1}{\left(\frac{\partial f}{\partial q_2}\right)^2}\right)dp_1$$

$$=\left(\frac{\left(\frac{\partial^2 f}{\partial q_1^2}-\frac{\partial^2 f}{\partial q_2\partial q_1}\frac{p_1}{p_2}\right)\frac{\partial f}{\partial q_2}-\left(\frac{\partial^2 f}{\partial q_1\partial q_2}-\frac{\partial^2 f}{\partial q_2^2}\frac{p_1}{p_2}\right)\frac{\partial f}{\partial q_1}}{\left(\frac{\partial f}{\partial q_2}\right)^2}p_2\right)dq_1$$

$$+\left(\frac{\frac{\partial^2 f}{\partial q_2\partial q_1}\frac{\partial f}{\partial q_2}-\frac{\partial^2 f}{\partial q_2^2}\frac{\partial f}{\partial q_1}}{\left(\frac{\partial f}{\partial q_2}\right)^2}\right)dT$$

$$+\left(\frac{\left(\frac{\partial^2 f}{\partial q_2^2}\frac{\partial f}{\partial q_1}-\frac{\partial^2 f}{\partial q_2\partial q_1}\frac{\partial f}{\partial q_2}\right)\left(\frac{T-p_1q_1}{p_2}\right)}{\left(\frac{\partial f}{\partial q_2}\right)^2}+\frac{\frac{\partial f}{\partial q_1}}{\frac{\partial f}{\partial q_2}}\right)dp_2. \tag{17}$$

We can present Eq. (17) as

$$a_1dp_1 = a_2dq_1 + a_3dT + a_4dp_2, \quad a_1 > 0,\ a_2 < 0,\ a_3 > 0, \tag{18}$$

where by a_i, $i = 1, \ldots, 4$ are denoted the coefficients of the differentials of which a_4 is of ambiguous sign. From (18) we can solve

$$\left.\frac{\partial p_1}{\partial q_1}\right|_{dT=dp_2=0}=\frac{a_2}{a_1}<0, \quad \left.\frac{\partial p_1}{\partial T}\right|_{dq_1=dp_2=0}=\frac{a_3}{a_1}>0, \quad \left.\frac{\partial p_1}{\partial q_2}\right|_{dT=dq_1=0}=\frac{a_4}{a_1},$$

where the sign of the last partial is ambiguous. Because p_1, h_1 both have unit $\$/kg$, they can be measured on the same coordinate axis. The slope $\frac{\partial p_1}{\partial q_1} = \frac{a_2}{a_1} < 0$ of the demand relation (16) in coordinate system $(q_1, \$/kg)$ deviates from that of the willingness-to-pay: $\frac{\partial h_1}{\partial q_1} = a_2 < 0$. Because $a_1 > 1$, the latter of the curves is steeper. The reason for this is the income effect a change in price has on the willingness-to-pay. If the price of food decreases, a consumer's utility maximizing flow of food consumption increases. However, a price decrease raises the real budgeted funds of the consumer and moves his willingness-to-pay relation away from the origin. A price increase analogously moves the willingness-to-pay relation toward the origin.

Equations (12) and (16) give similar results concerning how quantities q_1, T, p_2 affect the optimal flow of food consumption of the consumer, and they both are useful. The demand relation is estimable from the real world by statistical methods with observed prices and consumption flows, and the willingness-to-pay relation can be quantified by making a questionnaire.

Example 1. Let the weekly utility function of a consumer be $u = aq_1q_2$, where a with unit $(ut \times week)/(kg \times h)$ is a positive constant and the budget equation as earlier. With this utility function the marginal utilities are

$$\frac{\partial u}{\partial q_1} = aq_2 > 0, \quad \frac{\partial u}{\partial q_2} = aq_1 > 0,$$

and the sufficient condition for maximal utility holds,

$$\frac{\partial^2 u}{\partial q_1^2} = \frac{\partial^2 u}{\partial q_2^2} = 0, \quad \frac{\partial^2 u}{\partial q_1 \partial q_2} = \frac{\partial^2 u}{\partial q_2 \partial q_1} = a > 0.$$

Solving q_2 from the budget equation and setting in the utility function gives

$$u = \frac{aq_1}{p_2}\left(T - p_1q_1\right).$$

The necessary condition for optimization is then

$$\frac{du}{dq_1} = 0 \quad \Leftrightarrow \quad \frac{a}{p_2}\left(T - 2p_1q_1\right) = 0 \quad \Rightarrow \quad q_1^* = \frac{T}{2p_1} \quad \Leftrightarrow \quad p_1 = \frac{T}{2q_1}, \qquad (19)$$

and the sufficient condition for maximum holds: $d^2u/dq_1^2 = -2ap_1/p_2 < 0$. We call function q_1^* this *consumer's demand function of food, and the last form of the equation his inverse demand function of food*. Price p_2 does not affect q_1^* in this case which result is caused by the assumed form for the utility function. The consumer's willingness-to-pay for food is

$$h_1 = p_2\frac{\frac{\partial u}{\partial q_1}}{\frac{\partial u}{\partial q_2}} \quad \text{where} \quad \frac{\partial u}{\partial q_1} = aq_2, \quad \frac{\partial u}{\partial q_2} = aq_1.$$

Thus

$$h_1 = \frac{p_2q_2}{q_1} = \frac{T}{q_1} - p_1; \qquad (20)$$

the latter form is obtained by substituting the budget equation $p_2q_2 = T - p_1q_1$ in (20). Another way to derive the willingness-to-pay is to divide $\frac{\partial u}{\partial q_1} = aq_2$ by $\frac{\partial u}{\partial T} = aq_1/p_2$. In the optimum $h_1 = p_1$. $\diamond$

Example 2. Let a consumer's weekly utility function be

$$u = A(aq_1)^c(bq_2)^{1-c}, \qquad (21)$$

where the quantities are as earlier, constants $A, a, b > 0$ have units $ut/week$, $week/kg$, $week/h$, respectively, and $0 < c < 1$ is a pure number. Utility is thus measured in units $ut/week$, and the terms in braces are dimensionless as they should for dimensional consistency. Marginal utilities of the two goods with units ut/kg, ut/h, respectively, are

$$\frac{\partial u}{\partial q_1} = Aac(aq_1)^{c-1}(bq_2)^{1-c} > 0, \tag{22}$$

$$\frac{\partial u}{\partial q_2} = Ab(1-c)(aq_1)^c(bq_2)^{-c} > 0, \tag{23}$$

and the second order partials are:

$$\begin{aligned} \frac{\partial^2 u}{\partial q_1^2} &= Aa^2c(c-1)(aq_1)^{c-2}(bq_2)^{1-c} < 0, \\ \frac{\partial^2 u}{\partial q_2^2} &= -Ab^2c(1-c)(aq_1)^c(bq_2)^{-c-1} < 0, \\ \frac{\partial^2 u}{\partial q_2 \partial q_1} &= Aabc(1-c)(aq_1)^{c-1}(bq_2)^{-c} > 0. \end{aligned}$$

Marginal utilities are thus decreasing and the unique second order cross partial is positive; thus the sufficient condition for maximal utility holds. Substituting the earlier assumed budget equation in the utility function, we get

$$u = A(aq_1)^c \left(\frac{b[T-p_1q_1]}{p_2}\right)^{1-c}.$$

The necessary condition for the consumer's optimum is

$$\begin{aligned} \frac{du}{dq_1} = 0 \quad \Leftrightarrow \quad & Aca(aq_1)^{c-1}\left(\frac{b[T-p_1q_1]}{p_2}\right)^{1-c} \\ -A(1-c)(aq_1)^c & \left(\frac{b[T-p_1q_1]}{p_2}\right)^{-c}\frac{bp_1}{p_2} = 0. \end{aligned} \tag{24}$$

From (24) we get the consumer's demand and inverse demand functions of food as

$$q_1^* = \frac{cT}{p_1} \quad \Leftrightarrow \quad p_1 = \frac{cT}{q_1}. \tag{25}$$

An increase in T increases and an increase in p_1 decreases the consumer's optimal flow of food consumption q_1^*. Price p_2 does not affect q_1^* also in this case. If we multiply the first order condition (24) by factor

$$\frac{(aq_1)^{-c}\left(\frac{b[T-p_1q_1]}{p_2}\right)^c p_2}{Ab(1-c)} > 0,$$

we get

$$\frac{c}{1-c}\left(\frac{T}{q_1}-p_1\right)-p_1=0$$

where $h_1=\frac{c}{1-c}\left(\frac{T}{q_1}-p_1\right)$ is the consumer's willingness-to-pay for food. Notice that we could have derived the willingness-to-pay also as

$$h_1=p_2\frac{\frac{\partial u}{\partial q_1}}{\frac{\partial u}{\partial q_2}}=\frac{cp_2q_2}{(1-c)q_1},$$

where $\frac{\partial u}{\partial q_1}$, $\frac{\partial u}{\partial q_2}$ are as in (22) and (23), respectively, and substituting there $p_2q_2=T-p_1q_1$ from the budget equation.

Solving the budget equation for q_1, substituting this in the utility function and optimizing with respect to q_2, we get the optimal consumption flow of playing video games as $q_2^*=(1-c)T/p_2$ $(h/week)$. Another way to get this result is to substitute q_1^* in the budget equation and solve it for q_2. $\diamond$

Assuming the following values for the constants $c=0.7$, $T=100$, we can present the demand and willingness-to-pay -relations in Examples 1, 2 with two values for p_1: $p_{10}=10$ and $p_{11}=20$. The functions in Example 1 are presented in Figure 1 and those in Example 2 in Figure 2. Notice that the demand relation (the thick curve) is graphed in both figures in the form of inverse demand. Figures 1, 2 show how the demand and willingness-to-pay relations are related to each other. Both are decreasing with increasing flow of food consumption, and the demand relation stays constant with a price change while the willingness-to-pay relation moves so that the two curves cross each other at current price.

Figure 1. The demand and two willingness-to-pay relations of food

Figure 2. The demand and two willingness-to-pay relations of food

The optimal flow of food consumption of this consumer can be presented graphically as the crossing point of the horizontal line representing the price of food and the demand relations in (19) and (25). In these points, the willingness-to-pay and demand schedules cross too, and they both define the same optimal flow of food consumption q_1^*, see Figures 1, 2.

4.2 Newtonian Theory of a Consumer

The dynamic consumer behavior studied in the previous section can be modelled mathematically as follows. We set $q_1'(t)$ to depend positively on quantity $\frac{du}{dq_1}$ so that $q_1'(t)=0$ when $\frac{du}{dq_1}=\frac{\partial f}{\partial q_1}-\frac{\partial f}{\partial q_2}\frac{p_1}{p_2}=0$. This corresponds to

$$q_1'(t)=g(F_1),\quad g'(F_1)>0,\quad g(0)=0,\quad F_1=\frac{du}{dq_1}, \tag{26}$$

where g is a function with the above characteristics. The first order Taylor series approximation of function g in the neighborhood of the optimum point $F_1 = \frac{\partial f}{\partial q_1} - \frac{\partial f}{\partial q_2}\frac{p_1}{p_2} = 0$ is

$$g(F_1) = g(0) + g'(0)(F_1 - 0) + \epsilon = g'(0) \times F_1 + \epsilon.$$

Assuming that $\epsilon = 0$ we can approximate Eq. (26) as

$$q_1'(t) = g'(0) \times F_1 \;\Leftrightarrow\; q_1'(t) = g'(0) \times \left(\frac{\partial f}{\partial q_1} - \frac{\partial f}{\partial q_2}\frac{p_1}{p_2}\right) \tag{27}$$

where $g'(0) > 0$ is a constant. The unit of $q_1'(t)$ is $kg/week^2$, that of $\frac{\partial f}{\partial q_1} - \frac{\partial f}{\partial q_2}\frac{p_1}{p_2}$ is ut/kg and the unit of $g'(0)$ equals with that of $g'(F_1) = dq_1'(t)/dF_1$ which is $(kg/week)^2/ut$. Eq. (27) is thus dimensionally homogeneous.

Now $q_1'(t)$ is the instantaneous acceleration of food consumption of the consumer. If the reason $\frac{\partial f}{\partial q_1} - \frac{\partial f}{\partial q_2}\frac{p_1}{p_2}$ for this acceleration is named as the *force acting upon the food consumption of this consumer*, we can denote $g'(0) = 1/m_1$ and name the positive constant m_1 as the *inertial 'mass' of food consumption of this consumer*. Equation (27) is then of the same form as the Newton's equation of motion, $a = (1/m) \times F \Leftrightarrow F = ma$, where a is acceleration, F force and m the mass of the moving particle.

'Mass' m_1 is the ratio between force and acceleration and it measures the sensitivity of the flow of food consumption of this consumer with respect to the force. The factors affecting m_1 are those which slow down changes in the flow of food consumption of this consumer; limited knowledge of compensating goods, time to find such goods etc. The inertial 'mass' of food consumption can be measured via the force and acceleration as $m_1 = \left(\frac{\partial f}{\partial q_1} - \frac{\partial f}{\partial q_2}\frac{p_1}{p_2}\right)/q_1'(t)$ when these quantities are known and deviate from zero. This corresponds to the definition of *inertial mass* in physics.

In all economic behavior, various inertial factors exist. For example, being habited in a good makes us reluctant to change it. Practising new things is many times repulsive even though we know we would gain from that. Various kinds of costs may also be related to a consumer's change of his bundle of consumption flows of goods, such as changing the nearest grocery store to a supermarket further away. Due to these reasons, the bundle of consumption flows of goods of a consumer may stay constant even though he directs non-zero forces upon his consumption of some goods. This phenomena can be added in the model in the form of *static friction*.

It is common in economics to talk about adjustment or transaction costs instead of static friction. Static friction is, however, a more general concept which contains also other factors resisting changes than the costs related to them. When we add static friction in the model we can explain by it that many times a consumer changes his bundle of consumption flows of goods only when the reasons become compelling enough. This way obtained model for dynamic consumer behavior is

$$m_1 q_1'(t) = \frac{\partial f}{\partial q_1} - \frac{\partial f}{\partial q_2}\frac{p_1}{p_2} + F_{S1}, \tag{28}$$

where the static friction force with unit ut/kg is denoted by F_{S1}.

Static friction F_{S1} contains factors that resist changes in the consumer's food consumption not included in his utility function and budget equation: laziness, stubborn habits, costs and trouble from changing food consumption etc. Measuring the static friction of a consumer requires the measuring of these factors and the definition of a weighted average of them with unit ut/kg. This, however, is omitted and static friction is treated as an unknown quantity the numerical value of which can be estimated by Eq. (28).

According to Eq. (28), $q_1'(t) > 0$ when $\frac{\partial f}{\partial q_1} - \frac{\partial f}{\partial q_2}\frac{p_1}{p_2} + F_{S1} > 0$ and vice versa. Further, $F_{S1} < 0$ when $\frac{\partial f}{\partial q_1} - \frac{\partial f}{\partial q_2}\frac{p_1}{p_2} > 0$ and vice versa, and $|F_{S1}| \leq |\frac{\partial f}{\partial q_1} - \frac{\partial f}{\partial q_2}\frac{p_1}{p_2}|$. The consumer thus changes his flow of food consumption only if the net benefit from this exceeds his static friction. Static friction does not affect the dynamics of food consumption after the adjustment has began, that is, after the active force component has exceeded the static friction. Static friction only explains that the flow of food consumption may not always be changed when the active force component deviates from zero.

Example 3. Let the utility function of a consumer be $u = aq_1q_2$ where a with unit $(ut \times week)/(kg \times h)$ is a positive constant and the budget equation as earlier. This functional form is assumed because it gives a simple form for the Newtonian equation of food consumption. If we had applied, for example, function (21) for utility, a quite complicated equation would result.

Solving the budget equation with respect to q_2 and substituting in the utility function, we get

$$u = \frac{aq_1}{p_2}\left(T - p_1q_1\right).$$

The force acting upon the food consumption of this consumer is then

$$\frac{du}{dq_1} = \frac{a}{p_2}\left(T - 2p_1q_1\right).$$

The Newtonian equation of food consumption with this force is

$$m_1 q_1'(t) = \frac{du}{dq_1} \quad \Leftrightarrow \quad m_1 q_1'(t) = \frac{a}{p_2}\big(T - 2p_1q_1(t)\big). \tag{29}$$

The solution of this differential equation is

$$q_1(t) = \frac{T}{2p_1} + C_0 e^{-\frac{2ap_1}{p_2 m_1}t}, \tag{30}$$

where e is the base of the natural logarithm, $C_0 = q_1(0) - T/2p_1$ $(kg/week)$ the constant of integration and time t has unit $week$. According to (30), $q_1(t)$ approaches its optimal value $q_1^* = T/2p_1$ with time because the exponential term vanishes with $t \to +\infty$. The asymptotic equilibrium state thus corresponds to the zero force situation assumed in the neoclassical theory.

In this example, the force was presented in the form du/dq_1 and not in the form of $h_1 - p_1$. The latter form of the force would be

$$\frac{T}{q_1} - 2p_1,$$

and if the Newtonian equation of food consumption is constructed with this force, the following non-linear differential equation results

$$\frac{T}{q_1} - 2p_1 = m_{11} q_1'(t) \quad \Leftrightarrow \quad T - 2p_1 q_1 = m_{11} q_1 q_1'(t)$$

the solution of which is much more difficult; notice that the positive constant m_{11} deviates from that of m_1. Because quantities

$$\frac{a}{p_2}(T - 2p_1 q_1) \quad \text{and} \quad \frac{T}{q_1} - 2p_1,$$

are simultaneously positive and negative — they have equal zero points with positive values of q_1 — they both can be applied as the force acting upon food consumption of this consumer. The advantage of the former is a more simple Newtonian equation and that of the latter is measurability; it has unit $\$/kg$ while the former has ut/kg. Notice that force du/dq_1 was derived so that the budget constraint was included in the utility function. Without this, the derivative would not function as a force.

Substituting (30) in the budget equation, the consumer's weekly consumption of video games can be solved as

$$q_2(t) = \frac{T}{2p_2} - \frac{C_0 p_1}{p_2} e^{-\frac{2ap_1}{p_2 m_1}t}.$$

The asymptotic equilibrium thus corresponds to the consumer's optimal situation: $q_1^* = T/2p_1$, $q_2^* = T/2p_2$. $\diamond$

The dynamic consumer behavior presented in this section has still one advantage as compared with the static neoclassical framework. Because time is omitted from the static neoclassical analysis, in that framework we cannot study how a consumer's changing wealth with time affects his consumption. In the proposed framework, this can be done as follows. Suppose a consumer gains wealth so that he can steadily increase funds for his consumption. The budgeted funds for his weekly consumption are then a function of time, and we assume a linear form for the function: $T(t) = T_0 + bt$, where T_0, b are positive constants with units $\$/week$, $\$/week^2$, respectively, and time t has unit *week*. Assuming the utility function as in Example 3, the following Newtonian equation results

$$m_1 q_1'(t) = \frac{a}{p_2}\big(T(t) - 2p_1 q_1(t)\big) \quad \Leftrightarrow \quad m_1 q_1'(t) = \frac{a}{p_2}\big(T_0 + bt - 2p_1 q_1(t)\big).$$

The solution of this is

$$q_1(t) = \frac{2ap_1T_0 - bp_2m_1}{4ap_1^2} + \frac{bt}{2p_1} + C_1 e^{-\frac{2ap_1}{p_2m_1}t},$$

where C_1 $(kg/week)$ is the constant of integration. Now with $t \to \infty$, $q_1(t) \to \infty$ because even though the exponential term vanishes, the linear time trend $(b/2p_1)t$ increases without limit with time. Notice that b can be as small as we like, for example 0.1 ($\$/week^2$), which causes the weekly increase $b\Delta t$ $= 0.1$ $(\$/week^2) \times 1$ $(week) = 0.1$ $(\$/week)$ in budgeted funds. The time dependency in the budget equation means that a static optimum does not exist. Thus *we can model economic growth in the proposed framework which cannot be done in the neoclassical one.* This shows that the neoclassical framework is not general enough to cover all real world economic behavior.

5 Conclusions

We extended the static neoclassical theory of a consumer into a dynamic form consistent with the static analysis. In this, we defined the 'economic force' by which the consumer acts upon his consumption. An isomorphism between economic dynamics and classical mechanics was proposed, which gives the equilibrium and non-equilibrium analysis in a single framework. This is possible because both sciences assume causal relations between quantities and use differential equations to model these relations. If we forget the physical or economic content of the used quantities, we are left with purely mathematical equations. If then the equations have the same form, we can interpret the economic quantities in a physical way or vice versa. This has been demonstrated earlier. The mathematical form of Black-Scholes equation is identical with a specific heat flow equation in physics, Goodwin's growth model is identical with Lotka-Volterra equations in biology etc.

If the same mathematical model can be applied in different sciences, these phenomena must be of isomorphic structure. Finding such analogies may help in the modelling. Lucas (1988) writes: “A successful theory of economic development clearly needs, in the first place, mechanics that are consistent with sustained growth and with sustained diversity of income levels. ... so a useful theory needs also to capture some forces for change in these patterns, and a mechanics that permits these forces to operate”. We hope that our framework meets these requirements.

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Appendix: The Uniqueness of the Willingness-To-Pay

Suppose a consumer is consuming n different goods. The following Lagrangian function can then be written for his utility maximization problem

$$\max_{\mathbf{x},\lambda_1} L_1(\mathbf{x},\lambda_1), \quad L_1(\mathbf{x},\lambda_1) = u(\mathbf{x}) + \lambda_1 \left(T - \sum_{i=1}^{n} p_i x_i \right),$$

where the vector of consumption flows of the goods is $\mathbf{x} = (x_1, \ldots, x_n)$, $u(\mathbf{x})$ is the weekly utility function and the measurement units of the quantities are: u : *ut/week*, T : \$/*week*, x_i : (*the quantity unit of good i*)/*week*, p_i: \$/(*the quantity unit of good i*), λ_1 : *ut*/\$. These units make the function dimensionally well-defined and the unit of λ_1 is suitable for it to measure the marginal utility of budgeted funds in the optimum; see Varian (1992 p. 108).

The necessary conditions for utility maximization are

$$\begin{aligned} \frac{\partial L_1}{\partial x_i} &= 0 \quad \Leftrightarrow \quad \frac{\partial u}{\partial x_i} - \lambda_1 p_i = 0 \quad i = 1, \ldots, n, \\ \frac{\partial L_1}{\partial \lambda_1} &= 0 \quad \Leftrightarrow \quad T - \sum_{i=1}^{n} p_i x_i = 0. \end{aligned}$$

From these we can solve

$$p_i = \frac{1}{\lambda_1}\frac{\partial u}{\partial x_i} \quad \Leftrightarrow \quad \lambda_1 = \frac{1}{p_i}\frac{\partial u}{\partial x_i}, \quad i = 1, \ldots, n \quad \text{and} \quad T = \sum_{i=1}^{n} p_i x_i.$$

According to section 4, quantity $(1/\lambda_1)\partial u/\partial x_i$ measures the willingness-to-pay of the consumer for one unit of good i in the consumer's optimum, and it equals the unit price p_i, $i = 1, \ldots, n$.

Next we transform the utility function as $f(u)$, where $f : \mathbf{R} \to \mathbf{R}$ is any differentiable function with $f'(u) > 0$. The Lagrangian becomes then

$$\max_{\mathbf{x},\lambda_2} L_2(\mathbf{x},\lambda_2), \quad L_2(\mathbf{x},\lambda_2) = f(u(\mathbf{x})) + \lambda_2 \left(T - \sum_{i=1}^{n} p_i x_i \right).$$

Excluding utility, the measurement units of the quantities are those as above. The positive transformation of the utility function changes the measurement unit of utility so that now any change in utility gets a smaller or greater numerical value — depending on whether $f'(u)$ is under or above unity — compared with that in the first case. We can take care of this by giving a new measurement unit for utility called *utili*. This is analogous to that of measuring length first in units *inch* and then in units *metre*, or vice versa.

The necessary conditions for utility maximization are now

$$\begin{aligned}\frac{\partial L_2}{\partial x_i} &= 0 \quad \Leftrightarrow \quad f'(u)\frac{\partial u}{\partial x_i} - \lambda_2 p_i = 0, \quad i = 1, \ldots, n,\\ \frac{\partial L_2}{\partial \lambda_2} &= 0 \quad \Leftrightarrow \quad T - \sum_{i=1}^{n} p_i x_i = 0.\end{aligned}$$

From these we can solve

$$p_i = \frac{f'(u)}{\lambda_2}\frac{\partial u}{\partial x_i}, \quad i = 1, \ldots, n \quad \text{and} \quad T = \sum_{i=1}^{n} p_i x_i,$$

where now $(f'(u)/\lambda_2)\partial u/\partial x_i$ measures the consumer's willingness-to-pay for one unit of good i in the optimum. Because prices p_i, $i = 1, \ldots, n$ were assumed fixed, comparing the necessary conditions in both two cases we get

$$\frac{f'(u)}{\lambda_2}\frac{\partial u}{\partial x_i} = \frac{1}{\lambda_1}\frac{\partial u}{\partial x_i}, \quad i = 1, \ldots, n.$$

In the optimum, the consumer's willingness-to-pay for one unit of good i is thus independent of the chosen utility function, $i = 1, \ldots, n$. In the latter case, the numerical value of the marginal utility of budgeted funds adjusts with transformation $f : \mathbf{R} \to \mathbf{R}$ so that the above equation holds.

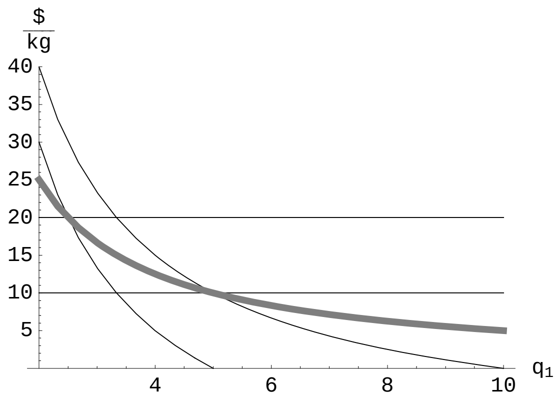

Figure 1. The demand and two willingness – to – pay relations of food

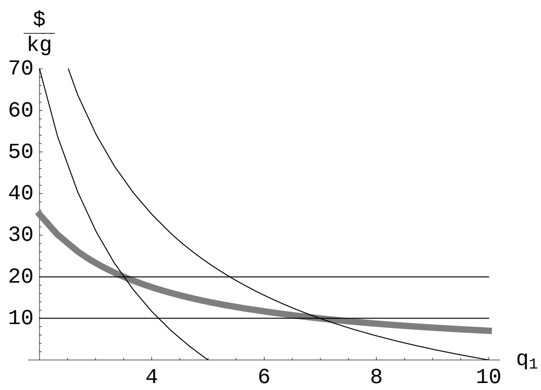

Figure 2. The demand and two willingness – to – pay relations of food