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YUEYANG ZHANG

DIFFERENCE NEVANLINNA THEORY AND ITS APPLICATIONS TO COMPLEX DIFFERENCE EQUATIONS

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Yueyang Zhang

DIFFERENCE NEVANLINNA THEORY AND ITS APPLICATIONS TO COMPLEX DIFFERENCE EQUATIONS

ACADEMIC DISSERTATION

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ABSTRACT

The survey part of this thesis introduces some new results concerning meromorphic solutions of complex difference equations. Before presenting these new results, classical Nevanlinna theory and its difference analogues are introduced. For some special functions of irregular growth, the lemma on the difference quotients is not applicable in a set of infinite linear measure. An application of the difference analogue of Clunie's lemma indicates that all meromorphic solutions of the equation: $f(z)^n + P_{n-1}(z, f(z)) = 0$, where $P_{n-1}(z, f(z))$ is a polynomial in $f(z)$ and its shifts of total degree at most $n-1$ with small coefficients, have hyper-order of growth at least 1. Moreover, difference Nevanlinna theory indicates some uniqueness results concerning meromorphic functions sharing values with their difference operators. Finally, in view of the difference analogue of the Painlevé property, a difference analogue of Steinmetz's generalization of Malmquist's 1913 classical theorem is provided by simplifying the first-order difference equation: $f(z+1)^n = R(z, f(z))$, where $R(z, f(z))$ is rational in both arguments, into a list of canonical equations.

MSC 2010: 30D35, 39A10, 39B32.
Keywords: Nevanlinna theory, difference analogue, complex difference equations, meromorphic solutions, growth, uniqueness theory.

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Joensuu, Aug. 08, 2018

Yueyang Zhang

LIST OF PUBLICATIONS

This thesis consists of the present review of the author's work in the field of complex difference equations and the following selection of the author's publications:

I R. Korhonen, K. Tohge, Y. Y. Zhang and J. H. Zheng, "A lemma on the difference quotients," submitted https://arxiv.org/abs/1806.00210

II Y. Y. Zhang, Z. S. Gao, J. L. Zhang, "On growth of meromorphic solutions of nonlinear difference equations and two conjectures of C.C. Yang," *Acta Mathematica Scientia* **36B** (2016), no. 1, 1–8.

III Z. S. Gao, R. Korhonen, J. L. Zhang, Y. Y. Zhang, "Uniqueness of meromorphic functions sharing values with their n^{th} order exact differences," to appear in *Analysis Mathematica*.

IV R. Korhonen, Y. Y. Zhang, "Existence of meromorphic solutions of first order difference equations," submitted https://arxiv.org/abs/1708.07647

Throughout the overview, these papers will be referred to by Roman numeral.

AUTHOR'S CONTRIBUTION

The publications selected in this dissertation are original research papers on difference Nevanlinna theory and its applications to complex difference equations.

The original versions of Papers II and III originate in the research work at Beihang University; significant revision to Paper III was made in Joensuu.

Papers I and IV were completed in Joensuu, all authors have made an equal contribution.

TABLE OF CONTENTS

1 Introduction

Nevanlinna theory studies the relations between three associated functions, namely *the proximity function* $m(r, f)$, *the counting function* $N(r, f)$ and *the characteristic function* $T(r, f)$. Since its birth in the 1920s, there have been numerous applications and analogies in many different fields of mathematics, such as differential equations, difference equations and number theory. The main results of Nevanlinna theory are known as the First Main Theorem and the Second Main Theorem.

One of the most important developments in Nevanlinna theory in recent years is the building of a difference analogue of Nevanlinna theory. In the 1980s, some mathematicians have applied classical Nevanlinna theory to study meromorphic solutions of complex difference equations and obtained several important results. In 2000, Ablowitz, Halburd and Herbst [1] suggested that the existence of at least one finite-order meromorphic solution of a difference equation is a good complex analytic analogue of the Painlevé property for complex differential equations. An ordinary differential equation is said to have the Painlevé property when all solutions are single-valued around all movable singularities. Their idea provided a solid base for the study of complex difference equations. In 2006, Halburd and Korhonen [26] implemented this idea on the second-order difference equation: $f(z+1) + f(z-1) = R(z, f)$, where $R(z, f)$ is rational in f with meromorphic coefficients, and showed that if this equation has an admissible meromorphic solution f of finite order, then it can be reduced to a list of canonical equations. These results appear to verify that the aforementioned approach by Ablowitz, Halburd and Herbst [1] is indeed a good complex analytic difference analogue of the Painlevé property.

To single out difference Painlevé equations from second-order difference equations, Halburd and Korhonen [23] proved a lemma on the difference quotients, which states that for finite-order meromorphic functions the quantity $m(r, f(z+c)/f(z))$ is small compared to $T(r, f)$ for all r outside a possible exceptional set of finite logarithmic measure. This lemma plays a key role in reducing the proof of [26, Theorem 1.1] to a problem involving only the relative frequencies of special values of the solution. At almost the same time, Chiang and Feng [12], independently, proved another form for estimate on the difference quotients without any exceptional set. The finite-order condition in the lemma on the difference quotients due to Halburd and Korhonen [23] was relaxed to hyper-order strictly less than one by Halburd, Korhonen and Tohge [30]. We show the necessity of the appearance of the exceptional set in the lemma on difference quotients due to Halburd, Korhonen and Tohge [30] by proving that for examples of functions of irregular growth, such as functions constructed by Miles [47], the estimate is not applicable in a set of infinite linear measure.

The lemma on the difference quotients has yielded a number of analogies of classic results for meromorphic functions, such as Clunie's theorem, Mohon'ko's theorem and the Second Main Theorem for the exact difference $\Delta f(z) = f(z+1) - f(z)$. Thus it is natural to apply these new tools to study the value distribution and growth of meromorphic solutions of complex difference equations. As an application of the difference analogue of Clunie's lemma [23, 30], we investigated the growth of meromorphic solutions of the general nonlinear difference equation: $f(z)^n + P_{n-1}(z, f(z)) = 0$, where $P_{n-1}(z, f(z))$ is a difference polynomial in $f(z)$ with meromorphic coefficients, and showed that any admissible solution f of this equation is of hyper-order at least 1. This indicates that the only one term with the highest degree in the difference equations can guarantee that all meromorphic solutions grow very rapidly.

Uniqueness theory studies the conditions under which two meromorphic solutions and their relations can be uniquely determined. Nevanlinna started this discussion by proving the famous five-value theorem. This theorem is then generalized to the case that if two non-constant meromorphic functions f and g share four distinct values CM, then f is a Möbius transformation of g. One of the main subjects in uniqueness theory is how meromorphic functions are connected to their derivatives by sharing certain values. With the development of difference Nevanlinna theory, it is natural to study the uniqueness problems of meromorphic functions sharing values with their shifts or n^{th} order differences $\Delta^n f(z)$. For a meromorphic function f of hyper-order less than 1, if f and its n^{th} order difference $\Delta^n f(z)$ share three distinct small periodic functions with period 1 CM, then f satisfies certain linear difference equations. Using difference Nevanlinna theory, we prove that $f(z) = \Delta^n f(z)$.

Malmquist [46] proved the classic result: If the first-order differential equation: $f' = R(z, f)$, where $R(z, f)$ is rational in both arguments, admits a transcendental meromorphic solution, then it reduces to the Riccati equation: $f' = a_2 f^2 + a_1 f + a_0$, with rational coefficients a_0, a_1 and a_2. Steinmetz [62], and Bank and Kaufman [3] proved the following generalization: If the equation: $(f')^n = R(z, f)$ has rational coefficients and a transcendental meromorphic solution, then by a suitable Möbius transformation it can be either mapped to the Riccati equation or to one of a list of five equations. In view of the difference Painlevé property proposed by Ablowitz, Halburd and Herbst [1], we present a natural difference analogue of this result by showing that if the difference equation: $f(z+1)^n = R(z, f)$, where the coefficients are rational, has a transcendental meromorphic solution f of hyper-order < 1, then either f satisfies a difference linear or Riccati equation or it reduces to a list of five equations consisting of four Fermat type difference equations and one equation which is a special case of the symmetric QRT map. All these equations indeed have meromorphic solutions of finite order.

The remainder of this survey is organized as follows. In Chapter 2, we first recall some basic notations and fundamental results of Nevanlinna theory. In Chapter 3, we discuss the lemma on the difference quotients and its applications to study the value distribution of meromorphic functions. In particular, we will discuss the necessity of the exceptional set. In Chapter 4, we first discuss the complex difference equations in general and consider the growth of solutions of a type of nonlinear difference equations using difference Nevanlinna theory. Second, we discuss how difference Nevanlinna theory can be used to study the uniqueness problems concerning meromorphic functions sharing values with their shifts or difference operators. Third, we discuss the existence of meromorphic solutions of complex difference equations. Finally, in Chapter 5, the essential contents of Papers I-IV are summarized.

2 Nevanlinna theory

In this section, we recall some basic notations and fundamental results of Nevanlinna theory [31,38,53].

2.1 CLASSICAL NEVANLINNA THEORY

Let $\mathcal{M}$ denote the field of meromorphic functions in the complex plane $\mathbb{C}$. For a function $f(z) \in \mathcal{M}$, Nevanlinna theory studies three associated real-valued functions: $m(r,f)$, $N(r,f)$ and $T(r,f)$. The *proximity function* $m(r,f)$ is defined to be

$$m(r,f) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta, \quad \log^+ x := \max(0, \log x).$$

In the following, we also use the notation $m(r,a,f)$, which means $m(r,f)$ when $a = \infty$ and $m(r, 1/(f-a))$ when $a \in \mathbb{C}$. The (integrated) *counting function* $N(r,f)$ is defined to be

$$N(r,f) := \int_0^r \frac{n(t,f) - n(0,f)}{t} dt + n(0,f)\log r,$$

where $n(r,f)$ is the number of poles (counting multiplicities) of $f(z)$ in the disc $\{z : |z| \le r\}$. Below, we also use the notation $N(r,a,f)$, which means $N(r,f)$ when $a = \infty$ and $N(r, 1/(f-a))$ when $a \in \mathbb{C}$. The *characteristic function* $T(r,f)$ is then defined as

$$T(r,f) := m(r,f) + N(r,f).$$

Using the characteristic function, the *order of growth* and *lower order of growth* of $f(z)$ are defined to be

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

and

$$\mu(f) = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r},$$

respectively. Furthermore, when $f(z)$ is of the order $\sigma(f) = \infty$, the *hyper-order* of $f(z)$ is defined to be

$$\varsigma(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

The fundamental results of Nevanlinna theory are known as the First Main Theorem and the Second Main Theorem.

Theorem 1 (The First Main Theorem)**.** *Let f be a non-constant meromorphic function. Then for any complex number $a \in \mathbb{C}$,*

$$T\left(r, \frac{1}{f-a}\right) = T(r,f) + O(1).$$

The lemma on the logarithmic derivatives is recognized as one of the most significant results in value distribution theory and plays a key role in proving the Second Main Theorem [52].

Lemma 1 (The lemma on the logarithmic derivatives)**.** *Suppose that f is meromorphic and non-constant in the complex plane. Then*

$$m\left(r, \frac{f'}{f}\right) = O(\log r)$$

as $r \to \infty$ when f is of finite order and

$$m\left(r, \frac{f'}{f}\right) = O(\log rT(r, f))$$

as $r \to \infty$ outside a possible set of finite linear measure when f is of infinite order.

The following is the Second Main Theorem.

Theorem 2 (The Second Main Theorem)**.** *Let f be a non-constant meromorphic function, $q \geq 1$ and $a_1, a_2, \ldots, a_q$ be distinct complex numbers. Then*

$$(q-1)T(r, f) \leq N(r, f) + \sum_{i=1}^{q} N\left(r, \frac{1}{f - a_i}\right) - N_1(r, f) + S(r, f),$$

where $N_1(r, f) := 2N(r, f) - N(r, f') + N(r, 1/f')$ is a positive quantity measuring the number of multiple a-points, and $S(r, f) = O(\log r)$, $r \to \infty$, when f is of finite order and $S(r, f) = O(\log rT(r, f))$, $r \to \infty$, outside a possible set of linear measure when f is of infinite order.

The Second Main Theorem is a deep generalization of Picard's theorem stating that a non-constant entire function can omit at most one finite value on $\mathbb{C}$. The Second Main Theorem can be simplified to the following form:

$$(q-2)T(r, f) \leq \sum_{i=1}^{q} \overline{N}(r, a_i, f) + S(r, f), \tag{2.1}$$

where $q \geq 1$, $\overline{N}(r, a, f)$ is the truncated counting function for a-points or poles. This inequality implies the following *defect relation*:

$$\sum_{a \in \mathbb{C} \cup \{\infty\}} (\delta(a, f) + \theta(a, f)) \leq \sum_{a \in \mathbb{C} \cup \{\infty\}} \Theta(a, f) \leq 2, \tag{2.2}$$

where the *deficiency* $\delta(a, f)$ and the *index of multiplicity* $\theta(a, f)$, $a \in \mathbb{C} \cup \{\infty\}$, are defined as

$$\delta(a, f) := 1 - \limsup_{r \to \infty} \frac{N(r, a, f)}{T(r, f)} = \liminf_{r \to \infty} \frac{m(r, a, f)}{T(r, f)},$$

and

$$\theta(a, f) := \liminf_{r \to \infty} \frac{N(r, a, f) - \overline{N}(r, a, f)}{T(r, f)},$$

respectively, and the quantity $\Theta(a, f)$ is defined as

$$\Theta(a, f) := 1 - \limsup_{r \to \infty} \frac{\overline{N}(r, a, f)}{T(r, f)}.$$

Obviously, $0 \leq \delta(a, f) \leq 1$, $0 \leq \theta(a, f) \leq 1$. Moreover, if the order of all zeros of $f(z) - a = 0$ is at least 2, then the value a is said to be a *completely ramified value* of $f(z)$. The defect relation (2.2) indicates directly that a non-constant meromorphic function $f(z)$ can have at most four completely ramified values over $\mathbb{C} \cup \{\infty\}$. By distinguishing between omitted values and non-omitted completely ramified values and noting that any omitted value of $f(z)$ is maximally deficient, i.e., $\delta(a, f) = 1$, a non-constant meromorphic function $f(z)$ can have at most two omitted values or one omitted value plus two completely ramified values over $\mathbb{C} \cup \{\infty\}$.

In the Second Main Theorem, the exceptional set associated with the error term $S(r, f) = O(\log rT(r, f))$ is of finite linear measure. But in the study of complex difference equations, the exceptional set we have to deal with is often a little larger. For convenience, we still use the notation $S(r, f)$ and from now on this notation denotes any quantity that satisfies the condition

$$\lim_{r\to\infty} \frac{S(r, f)}{T(r, f)} = 0,$$

outside of a possible exceptional set E with finite logarithmic measure, that is, $\int_E dr/r < \infty$. For a meromorphic function $c(z) \in \mathcal{M}$, if $c(z)$ satisfies $T(r, c(z)) = S(r, f)$, then $c(z)$ is said to be *small* or *slowing moving* compared to $f(z)$. For example, all rational functions are small with respect to any transcendental meromorphic function. Denote by $\mathcal{S}(f)$ the field of all small functions of $f(z)$, i.e.,

$$\mathcal{S}(f) = \{c(z) \in \mathcal{M} : T(r, c(z)) = S(r, f)\}.$$

Set $\hat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$. A meromorphic solution $f(z)$ of a differential or difference equation is called *admissible* if all coefficients of the equation are in $\mathcal{S}(f)$.

The three quantities $\delta(a, f)$, $\theta(a, f)$ and $\Theta(a, f)$ can be defined similarly for small functions of f. Nevanlinna suggested the question of whether or not the defect relation (2.2) remains valid if the sum is taken over $\hat{\mathcal{S}}(f)$. Yamanoi [66, Corollary 1] solved this problem completely by proving

Theorem 3 (The Second Main Theorem of Yanamoi)**.** *Let f be a non-constant meromorphic function on $\mathbb{C}$, and let $a_1, \ldots, a_q$ be distinct small meromorphic functions of f on $\mathbb{C}$. Then we have the second main theorem,*

$$(q - 2 - \varepsilon)T(r, f) \leq \sum_{i=1}^{q} \overline{N}(r, a_i, f), \quad \textit{for all} \quad \varepsilon > 0, \tag{2.3}$$

where $r \notin E$ for some exceptional set E with $\int_E d\log\log r < \infty$, and the defect relation

$$\sum_{i=1}^{q} (\delta(a_i, f) + \theta(a_i, f)) \leq 2.$$

In particular, when all a_i are rational functions, (2.3) holds outside an exceptional set of finite linear measure [67].

In what follows, we state that $c(z) \in \hat{\mathcal{S}}(f)$ is a *completely ramified small function* of $f(z)$ when $f(z) - c(z) = 0$ has at most $S(r, f)$ many simple roots and that $c(z) \in \hat{\mathcal{S}}(f)$ is a *Picard exceptional small function* of $f(z)$ when $N(r, c, f) = S(r, f)$. Obviously, a non-constant meromorphic function $f(z)$ can have at most two Picard exceptional small functions. Moreover, we have the following result.

Theorem 4. *A non-constant meromorphic function $f(z)$ can have at most four completely ramified small functions.*

An identity concerning the Nevanlinna characteristics introduced originally by Valiron [63], and generalized by Mohon'ko [48], is very useful in the study of complex differential and difference equations; for the proof, see also [38].

Theorem 5. *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f) = \frac{P(z,f)}{Q(z,f)} = \frac{\sum_{i=0}^{p} a_i(z) f^i}{\sum_{j=0}^{q} b_j(z) f^j},$$

such that the meromorphic coefficients $a_i(z)$, $b_j(z)$ satisfy

$$\begin{cases} T(r, a_i(z)) = S(r,f), & i = 0,1,\ldots,p, \\ T(r, b_i(z)) = S(r,f), & i = 0,1,\ldots,q, \end{cases}$$

we have

$$T(r, R(z,f)) = \max\{p,q\} T(r,f) + S(r,f).$$

Finally, when considering the meromorphic solution f of complex differential or difference equations, we sometimes need to make a transformation to the solution using some algebroid functions and achieve a situation where the considered function has some finite-sheeted branching. The classical version of Nevanlinna theory introduced above cannot be used to handle this situation. In this case, we need the Selberg-Ullrich theory, the algebroid version of Nevanlinna theory (see, for instance, [36]), which studies meromorphic functions on a finitely sheeted Riemann surface.

Definition 1. *An* algebroid function *is a n-valued function $f(z)$ defined by an irreducible equation*

$$F(z) = A_n(z) f(z)^n + A_{n-1}(z) f(z)^{n-1} + \ldots + A_1(z) f(z) + A_0(z) = 0, \tag{2.4}$$

where the coefficients are entire functions on the complex plane such that $A_0(z)A_n(z) \not\equiv 0$.

For the definitions of $m(r,f)$, $N(r,f)$ and $T(r,f)$ of an algebroid function f and the corresponding First Main Theorem and Second Main Theorem, we refer to [36]. In the proof of the main theorem of Paper IV, the algebroid functions we need to consider are either small functions with respect to an admissible meromorphic solution f of a difference equation or can be obtained from it by a Möbius transformation with small algebroid coefficients. Such functions could be described as "almost meromorphic" in the sense of Nevanlinna theory, since the presence of branch points actually only affects the small error term $S(r,f)$ in any of the estimates involving Nevanlinna functions.

3 Difference Nevanlinna theory

3.1 THE LEMMA ON THE DIFFERENCE QUOTIENTS

Lemma 1 is an important tool in analyzing the value distribution of entire and meromorphic solutions of differential equations [38]. For example, Yosida [72] used it to give a direct proof of the classic Malmquist theorem [46]. The Nevanlinna theoretic approach by Ablowitz, Halburd and Herbst [1] to study difference Painlevé equations leads to a need to find extensions of value distribution theory for difference operators. The lemma on the difference quotients for finite-order meromorphic functions was introduced in two independent studies, by Halburd and Korhonen [23,25], and by Chiang and Feng [12]. We recall the following version from [30] which holds for meromorphic functions of hyper-order less than 1.

Theorem 6 (see [30]). *Let $f(z)$ be a non-constant meromorphic function of hyper-order strictly less than 1, $\varepsilon > 0$ and $c \in \mathbb{C}$. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\varsigma-\varepsilon}}\right), \tag{3.1}$$

for all r outside a possible exceptional set of finite logarithmic measure.

This lemma is a natural difference analogue of the lemma on the logarithmic derivatives, i.e., Lemma 1. For a constant $\eta \in \mathbb{C} \setminus \{0\}$, define the exact differences of $f(z)$ by $\Delta_\eta f(z) = f(z+\eta) - f(z)$ and $\Delta_\eta^n f(z) = \Delta_\eta(\Delta_\eta^{n-1} f(z)) = \sum_{i=0}^{n} (-1)^{n-i}\binom{n}{i} f(z + i\eta)$, where $n \geq 2$ is an integer. Moreover, we use the usual notation $\Delta^n f(z)$ when $\eta = 1$. Denote the field of η-periodic meromorphic functions defined in $\mathbb{C}$ of hyper-order strictly less than 1 by $\mathcal{P}_\eta^1$. It can be easily derived from (3.1) that [25]: Let $n \geq 1$ be an integer, and let $f(z)$ be a meromorphic function of hyper-order strictly less than 1. If $a \in \mathcal{P}_\eta^1 \cap \mathcal{S}(f)$, then

$$m\left(r, \frac{\Delta_\eta^n f(z)}{f(z) - a(z)}\right) = S(r,f).$$

In the proof of Theorem 6, the first step to obtain the estimate (3.1) is to get

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|,f)}{r^{1-\varsigma-\varepsilon}}\right), \tag{3.2}$$

which holds for all r outside an exceptional set of finite logarithmic measure. Then (3.1) is obtained by combining (3.2) with the following Borel type growth lemma.

Lemma 2 (see [30]). *Let $T : [0,\infty) \to (0,\infty)$ be a non-decreasing continuous function and let $s \in (0,\infty)$. If the hyper-order of T is strictly less than 1, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{r} = \varsigma < 1, \tag{3.3}$$

and $\delta \in (0, 1-\varsigma)$, then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right), \tag{3.4}$$

where r runs to infinity outside of a set of finite logarithmic measure.

It is shown in [19, p. 66] (see also [1] or [68]) that the following double inequalities

$$\begin{aligned}(1+o(1))T(r-|c|, f(z)) \leq T(r, f(z+c)) \leq (1+o(1))T(r+|c|, f(z)),\\ (1+o(1))N(r-|c|, f(z)) \leq N(r, f(z+c)) \leq (1+o(1))N(r+|c|, f(z)),\end{aligned} \tag{3.5}$$

hold as $r \to \infty$. By combining (3.5) and (3.4), we immediately observe that for a non-constant meromorphic function f of hyper-order less than 1,

$$\begin{aligned}T(r, f(z+c)) &= T(r, f(z)) + S(r, f),\\ N(r, f(z+c)) &= N(r, f(z)) + S(r, f).\end{aligned} \tag{3.6}$$

The estimate on difference quotients given by Chiang and Feng [12] for finite-order meromorphic functions takes the following form.

Theorem 7 (see [12]). *Let c_1, c_2 be two complex numbers such that $c_1 \neq c_2$ and let $f(z)$ be a finite-order meromorphic function. Let σ be the order of $f(z)$, then for each $\varepsilon > 0$, we have*

$$m\left(r, \frac{f(z+c_1)}{f(z+c_2)}\right) = O(r^{\sigma-1+\varepsilon}). \tag{3.7}$$

Chiang and Feng [12] also proved an estimate on the integrated counting function $N(r, f(z+c))$. The exponent of convergence of poles of a meromorphic function f is defined as

$$\lambda\left(\frac{1}{f}\right) := \limsup_{r\to\infty} \frac{\log N(r, f)}{\log r}.$$

Theorem 8 (see [12]). *Let $f(z)$ be a meromorphic function with exponent of convergence of poles $\lambda(1/f) = \lambda < \infty$, $c \neq 0$ fixed, then for any $\varepsilon > 0$,*

$$N(r, f(z+c)) = N(r, f) + O(r^{\lambda-1+\varepsilon}) + O(\log r). \tag{3.8}$$

Theorems 7 and 8 imply the following precise asymptotic relation between the characteristic functions $T(r, f(z+c))$ and $T(r, f)$.

Theorem 9 (see [12]). *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f)$, $\sigma < \infty$, and let c be a fixed nonzero complex number, then for any $\varepsilon > 0$, we have*

$$T(r, f(z+c)) = T(r, f) + O(r^{\sigma-1+\varepsilon}) + O(\log r). \tag{3.9}$$

Chiang and Feng [13, 14] and, independently, Bergweiler and Langley [7] have obtained Wiman-Valiron type estimates for difference quotients in the case of order < 1 meromorphic functions. These results extend a Wiman-Valiron method for differences due to Ishizaki and Yanagihara [35].

The lemma on the difference quotients was recently extended to include meromorphic functions with minimal type by Zheng and Korhonen [75]. In Paper I, we extend Lemma 2 to the case that the function $T(r)$ satisfies

$$\limsup_{r\to\infty} \frac{h(r)h(rh(r))\log T(r)}{r} = \zeta, \tag{3.10}$$

where $\zeta \in [0,\infty)$ and $h : [r_0,\infty) \to (0,\infty)$ is an increasing function such that

$$\int_{r_0}^{\infty} \frac{1}{th(t)}\,dt$$

converges. For the function $T(r)$ and any finite number $s > 0$, we have

$$T(r+s) = T(r) + (\zeta + o(1))\left(\frac{T(r)}{h(r)}\right), \tag{3.11}$$

where r runs to infinity outside of a set of finite logarithmic measure. Note that, for a given $\varepsilon > 0$, we may specialize $h(r)$ above to be $h(r) = (\log r)^{1+\varepsilon}$, where $r \in [r_0,\infty)$ and $r_0 > e$. With this specialization, we extend the estimate (3.1) to a slightly more general case of meromorphic functions $f(z)$ such that $\log T(r,f) \leq ar/(\log r)^{2+\nu}$, $a,\nu > 0$, for all sufficiently large r. We prove that for such meromorphic functions,

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{(\log r)^{\nu-\varepsilon}}\right), \tag{3.12}$$

where r runs to infinity outside of a set of finite logarithmic measure.

As we have seen in Theorem 6, there is an exceptional set in the estimate (3.1). It is natural to ask whether the appearance of this exceptional set is necessary? With a superficial observation, in estimates of the type (3.7) the exceptional set does not seem to be present. However, for functions of irregular growth whose lower order $\mu(f)$ satisfies $\mu(f) \leq \sigma(f) - 1$, the error term on the right-hand side of (3.7) is, in fact, bigger than the characteristic $T(r,f)$ for a large part of the positive real line. In Paper I, it is shown that the exceptional set in the estimate (3.1) must be of infinite linear measure for some special functions of irregular growth, for example, for functions constructed by Gol'dberg [6] and by Miles [47]. In particular, it is shown that there is an infinite sequence of r_n in the set along which $m(r_n, f(z+c)/f(z))$ is not small compared to $T(r_n,f)$ for entire functions constructed by Miles [47]. Note that the function $m(r, f(z+c)/f(z))$ can grow as rapidly as $T(r,f)$ for all $r \in [0,\infty)$ when f has hyper-order at least 1, for example, for $f(z) = \exp(e^z)$.

3.2 ANALOGOUS RESULTS OF NEVANLINNA THEORY

In this section, we present some difference analogues of the results from Nevanlinna theory, which are natural consequences of Theorem 6. These include the difference analogues of Clunie's theorem [17], A. Z. Monkon'ko and V. D. Monkon'ko's theorem [49], the Second Main Theorem and the classic Borel's lemma on entire functions without zeros [40]. Before stating these results, we define a *difference polynomial* in $f(z)$ as a function which is a polynomial in $f(z+c_j)$, where c_j, $j = 1,\ldots,n$, are constants, with meromorphic coefficients that are small compared to $f(z)$.

Theorem 10 (see [23,30])**.** *Let $f(z)$ be a non-constant meromorphic solution of*

$$f(z)^n P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are difference polynomials in $f(z)$ and the degree of $Q(z,f)$ as a polynomial in $f(z)$ and its shifts is at most n. If the hyper-order of f is less than 1, then

$$m(r, P(z,f)) = S(r,f),$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Laine and Yang [39] proved a more general form of the difference analogue of Clunie's theorem by showing that the conclusion of Theorem 10 still holds when the term $f(z)^n$ is replaced by a difference polynomial in $f(z)$ containing only one term of maximal degree in $f(z)$ and its shifts.

The following theorem is a difference analogue of A. Z. Monkon'ko and V. D. Monkon'ko's theorem [49] for differential equations.

Theorem 11 (see [23,30])**.** *Let $f(z)$ be a non-constant meromorphic solution of*

$$P(z, f) = 0,$$

where $P(z, f)$ is a difference polynomial in $f(z)$. If the hyper-order of f is less than 1 and $P(z, a) \not\equiv 0$ for a slowly moving target a, then

$$m\left(r, \frac{1}{f-a}\right) = S(r, f),$$

for all r outside of a possible exceptional set with finite logarithmic measure.

Theorems 10 and 11 are presented in the most general form and thus can in many cases be used to study the value distribution of meromorphic solutions of complex difference equations.

Define a *differential-difference polynomial* in f as a finite sum of the products of f, derivatives of f and of their shifts, with all the coefficients of these monomials being small functions of f. Then the conclusions of Theorem 10 and 11 both hold true when $P(z, f)$ is replaced by a differential-difference polynomial in f of hyper-order less than 1.

Following the classical way, Halburd and Korhonen [25,30] proved a difference analogue of the Second Main Theorem for the exact difference $\Delta_\eta = f(z+\eta) - f(z)$.

Theorem 12 (The Second Main Theorem)**.** *Let $\eta \in \mathbb{C} \setminus \{0\}$, and let f be a meromorphic function of hyper-order less than 1 such that $\Delta_\eta f \not\equiv 0$. Let $q \geq 2$, and let $a_1, a_2, \ldots, a_q$ be distinct meromorphic periodic functions with period η such that $a_k \in \mathcal{S}(f)$ for all $k = 1, \ldots, q$. Then*

$$m(r, f) + \sum_{k=1}^{q} m\left(r, \frac{1}{f-a_k}\right) \leq 2T(r, f) - N_{pair}(r, f) + S(r, f),$$

where

$$N_{pair}(r, f) = 2N(r, f) - N(r, \Delta_\eta f) + N\left(r, \frac{1}{\Delta_\eta f}\right),$$

and the exceptional set associated with $S(r, f)$ is of at most finite logarithmic measure.

The Second Main Theorem for the exact difference $\Delta_\eta f$ above has many implications, such as new interpretations of Nevanlinna's defect relation, Picard's theorem and Nevanlinna's five-value theorem [25]. Halburd, Korhonen and Tohge [30] then considered a more general case of the holomorphic curves and developed the difference analogue of Cartan's generalization of the Second Main Theorem. The above Theorem 12, for constant targets, follows directly from [30, Theorem 3.1]. To prove [30, Theorem 3.1], Halburd, Korhonen and Tohge [30] developed a number of useful tools which can be used in the study of complex difference equations. Below we only recall Lemma 3 from [30], which is a difference analogue of the classic

Borel's lemma (see, e.g., [40]) for entire functions with no zeros. We say the zero z_0 of an entire function $g(z)$ with order $i \geq 1$ is *forward invariant* with respect to the translation $\tau(z) = z + \eta$ when z_0 is also a zero of $g(z + \eta)$ with order j and $j \geq i$. For example, all the zeros of an entire function with period η are forward invariant with respect to the translation $\tau(z) = z + \eta$.

Lemma 3 (see [30])**.** *Let $\eta \in \mathbb{C}$, and $g_0, \ldots, g_n$ be entire functions such that $\varsigma(g_i) < 1$, $i = 0, \ldots, n$, and such that all zeros of $g_0, \ldots, g_n$ are forward invariant with respect to the translation $\tau(z) = z + \eta$. If $g_i/g_j \notin \mathcal{P}^1_\eta$ for all $i, j \in \{0, \ldots, n\}$ such that $i \neq j$, then $g_0, \ldots, g_n$ are linearly independent over $\mathcal{P}^1_\eta$.*

Finally, we remark that there are some other analogous versions of the lemma on the logarithmic derivatives. For example, the q-shift version for zero-order meromorphic functions in [4], the tropical version for meromorphic functions of hyper-order less than 1 in [29], the Askey-Wilson logarithmic difference version for meromorphic functions of finite logarithmic order by Chiang and Feng [15] and the logarithmic Wilson differences version for finite-order meromorphic functions by Cheng and Chiang [10]. Each of these analogues of the lemma on the logarithmic derivatives has led to the establishment of a type of Second Main Theorem. There are also corresponding new interpretations on Nevanlinna's classic results.

4 Meromorphic solutions of complex difference equations

4.1 NONLINEAR DIFFERENCE EQUATIONS

Let $p, q \in \mathbb{N}$. In the following, the notation $R(z, f(z))$ will always mean that

$$R(z, f(z)) = \frac{a_0(z) + a_1(z)f(z) + \ldots + a_p(z)f(z)^p}{b_0(z) + b_1(z)f(z) + \ldots + b_q(z)f(z)^q},$$

where the coefficients a_i, b_j are small functions of $f(z)$.

According to a classic result by Malmquist [46], if the first-order differential equation

$$f' = R(z, f),$$

where $R(z, f)$ is rational in both arguments, has a transcendental meromorphic solution, then (4.1) reduces to the Riccati equation

$$f' = a_2 f^2 + a_1 f + a_0$$

with rational coefficients a_0, a_1 and a_2. Yosida [72,73] gave elegant alternative proofs of this theorem using tools from Nevanlinna theory. When considering the existence of meromorphic solutions of complex difference equations, the order of growth of the solutions is often concerned. For example, Yanagihara [68] proved that if the first-order equation

$$f(z+1) = R(z, f(z)), \tag{4.1}$$

where $R(z, f(z))$ is rational in both arguments, has a transcendental meromorphic solution of hyper-order strictly less than one, then $\deg_f(R(z, f(z))) = 1$ and thus (4.1) reduces to the difference Riccati equation. This is a natural difference analogue of Malmquist's result above on differential equations. In the second order case, Ablowitz, Halburd and Herbst [1] proved

Theorem 13 (see [1])**.** *If the second-order difference equation*

$$f(z+1) + f(z-1) = R(z, f(z)) \tag{4.2}$$

where $R(z, f(z))$ is rational in both arguments, admits a non-rational solution of finite order, then $\max\{p, q\} \leq 2$.

Theorem 14 (see [1])**.** *If the second-order difference equation*

$$f(z+1)f(z-1) = R(z, f(z)) \tag{4.3}$$

where $R(z, f(z))$ is rational in both arguments, admits a non-rational solution of finite order, then $\max\{p, q\} \leq 2$.

The above two equations (4.2) and (4.3) are closely related to difference Painlevé equations. In [1], the following result on a special case of (4.2) is also proved.

Theorem 15 (see [1]). *Any entire non-polynomial solution of the equation*

$$f(z+1)+f(z-1)=a(z)+b(z)f(z)+c(z)f(z)^2 \tag{4.4}$$

where a and b are polynomials and $c \neq 0$ is a constant, is of infinite order.

Laine and Yang [39] improved Theorem 15 by showing that any meromorphic solution of (4.4) is of infinite order when the coefficients a, b, c are all small functions of $f(z)$ and $c \not\equiv 0$. Theorem 13 is generalized in [33] to the higher-order case

$$\sum_{i=1}^{n} d_i(z)f(z+c_i) = R(z, f(z)),$$

where the coefficients are all small functions of $f(z)$: If the degree of $R(z, f(z))$ in $f(z)$ is greater than n, then $f(z)$ has infinite order. Using the estimate (3.9), Chiang and Feng [12] provided a direct proof for this result. On the other hand, Yang and Laine [71] studied the analogies between results on the existence of finite order entire solutions of the nonlinear differential-difference equation of the form

$$f(z)^n + L(z, f(z)) = h(z),$$

where $n \geq 2$ is an integer, $h(z)$ is a given non-vanishing meromorphic function of finite order and $L(z, f(z))$ is a linear differential-difference polynomial with small coefficients. They showed in [71, Theorem 2.6] that when $n \geq 4$, the above equation admits at most one admissible solution of finite order which is the same as $h(z)$. In Paper II, we investigate the growth of meromorphic solutions of the following general nonlinear difference equation

$$f(z)^n + P_{n-1}(z, f) = 0, \tag{4.5}$$

where $n \geq 2$ and $P_{n-1}(z, f)$ is a difference polynomial of degree at most $n-1$ in f. In this equation, the sole term whose degree is strictly higher than the rest will force all meromorphic solutions to have hyper-order at least 1.

Laine and Yang [71] also proposed two conjectures on nonlinear difference equations. For example, they conjecture that there exists no entire function of infinite order that satisfies a difference equation of the type

$$f(z)^n + q(z)f(z+1) = c \sin bz, \tag{4.6}$$

where q is a non-constant polynomial, b, c are nonzero constants and $n \geq 2$ is an integer. When $n = 3$, [71, Theorem 2.5] states that (4.6) can have finite order entire solutions only when $b = 3\pi n$ and $q(z)$ is a constant and $q = (-1)^{n+1}\frac{27}{4}c^2$. Because of the lack of efficient tools for meromorphic functions of hyper-order no less than 1, it is difficult to confirm this conjecture. But following the classic methods of Nevanlinna theory, we can study the value distribution of the solution of the above equation. In Paper II, it is shown that if $f(z)$ is an entire solution of (4.6) with infinite order and if $\varsigma(f) < \infty$, then for any entire function $d(z)$ of finite order, we have $\lambda(f-d) = \infty$.

4.2 UNIQUENESS PROBLEMS ON DIFFERENCE OPERATORS

Let f and g be two non-constant meromorphic functions in the complex plane $\mathbb{C}$, and a be a value on $\mathbb{C} \cup \{\infty\}$. We say that f and g share a CM (IM) provided that

f and g have the same a-points counting multiplicities (ignoring multiplicities). It is well known that if two non-constant meromorphic functions f and g share four distinct values CM, then f is a Möbius transformation of g. Rubel and Yang [60] initiated the study of entire functions sharing values with their derivatives, and proved that $f' \equiv f$ if a non-constant entire function f and its derivative f' share two distinct finite values CM. Mues and Steinmetz [50] and Gundersen [22] extended this conclusion to meromorphic functions.

Theorem 16 (see [22, 50])**.** *If a meromorphic function $f(z)$ and its derivative $f'(z)$ share two distinct values $a_1, a_2 \in \mathbb{C}$ CM, then $f' \equiv f$.*

Frank and Weißenborn [18] extended Theorem 16 by showing that the conclusion of Theorem 16 still holds when replacing f' with the n$^{\text{th}}$ derivative $f^{(n)}$, $n \geq 2$.

With the developments in difference Nevanlinna theory, it is natural to consider the unicity problems concerning meromorphic functions sharing values with their shifts or difference operators. For meromorphic functions sharing one value with their shifts, Heittokangas *et. al* [34] proved an analogue of the famous Brück conjecture [8]. The Brück conjecture state that if an entire function f shares one value a CM with f', then $f' \equiv f$. For meromorphic functions sharing two or more values with their shifts, Heittokangas *et. al* [32] proved the following shift analogue of Theorem 16 for small periodic functions of $f(z)$ using Theorem 12. Recall that $\mathcal{S}(f)$ denotes the field of all small functions of $f(z)$ and $\hat{\mathcal{S}}(f) = \mathcal{S}(f) \cup \{\infty\}$.

Theorem 17 (see [32])**.** *Let $f(z)$ be a meromorphic function of finite order, and let $\eta \in \mathbb{C}$. If $f(z)$ and $f(z+\eta)$ share three distinct functions $a, b, c \in \hat{\mathcal{S}}(f)$ with period η CM, then $f(z) = f(z+\eta)$ for all $z \in \mathbb{C}$.*

Heittokangas *et. al* [34] have improved the conclusion of Theorem 17 by replacing the condition 3 CM with 2 CM + 1 IM. Moreover, the conclusion of Theorem 17 still holds when f has hyper-order $\varsigma(f) < 1$. This can be seen by applying Theorem 6 for meromorphic functions of hyper-order strictly less than 1, and by following the proof of Theorem 17 in [32].

Theorem 17 and its generalizations give us a good idea on how the uniqueness theory of meromorphic functions works when comparing a meromorphic function with its shift. However, when looking for a difference analogue of the derivative, the exact difference operator is a more natural analogue than the shift operator. This raises the following question: does the conclusion of Theorem 17 still hold for meromorphic functions of hyper-order strictly less than 1 when replacing the shift $f(z+\eta)$ with the n^{th} order difference $\Delta_\eta^n f(z)$? In papers [9, 43–45, 74], the authors have proved some uniqueness theorems related to this question on finite order meromorphic functions sharing values CM with their differences $\Delta_\eta^n f(z)$.

In Paper III, we give a positive answer to the question posed above. This can be viewed as a difference analogue of the following result by Li [42, Theorem 1] on the uniqueness of meromorphic functions sharing small functions with their n$^{\text{th}}$ order derivatives.

Theorem 18 (see [42])**.** *Let $f(z)$ be a transcendental meromorphic function, a_1 and a_2 ($a_1, a_2 \neq \infty$) be two distinct meromorphic functions satisfying $T(r, a_j) = S(r, f)$, $j = 1, 2$, and let $n > 1$ be a positive integer. If f and $f^{(n)}$ share a_1 and a_2 CM, then $f^{(n)} \equiv f$.*

Li [42] also showed that Theorem 18 is not generally valid when $n = 1$.

4.3 FIRST-ORDER DIFFERENCE EQUATIONS

Global existence of large classes of meromorphic solutions is a rare property for a differential equation to have. In the first-order case, Malmquist's theorem mentioned in the beginning of Section 4.1 is a classic result. In the second-order case, Painlevé [55,56] and his colleagues classified all meromorphic solutions out of the class

$$f'' = F(z, f, f'),$$

where F is rational in f and f' and analytic in z, which has the Painlevé property. Here an ordinary differential equation is said to have the Painlevé property when all solutions are single-valued around all movable singularities. Generalizations of Malmquist's theorem for the equation

$$(f')^n = R(z, f), \qquad n \in \mathbb{N}, \tag{4.7}$$

have been given by Yosida [72] and Laine [37]. Steinmetz [62], and Bank and Kaufman [3] proved that if (4.7) has rational coefficients and a transcendental meromorphic solution, then by a suitable Möbius transformation, (4.7) can be either mapped to (4.1), or to one of the equations in the following list:

$$\begin{aligned}
(f')^2 &= a(f-b)^2(f-\tau_1)(f-\tau_2),\\
(f')^2 &= a(f-\tau_1)(f-\tau_2)(f-\tau_3)(f-\tau_4),\\
(f')^3 &= a(f-\tau_1)^2(f-\tau_2)^2(f-\tau_3)^2,\\
(f')^4 &= a(f-\tau_1)^2(f-\tau_2)^3(f-\tau_3)^3,\\
(f')^6 &= a(f-\tau_1)^3(f-\tau_2)^4(f-\tau_3)^5,
\end{aligned}$$

where a and b are rational functions, and $\tau_1, \ldots, \tau_4$ are distinct constants.

The existence of globally meromorphic solutions is somewhat more common in the case of difference equations, as compared to differential equations. For first-order difference equations, apart from the difference analogue of Malmquist's result given by Yanagihara [68], Shimomura [61] showed that the difference equation

$$f(z+1) = P(f(z)),$$

where $P(f(z))$ is a polynomial in $f(z)$ with constant coefficients, always has a non-trivial entire solution. On the other hand, Yanagihara [68] demonstrated that the difference equation

$$f(z+1) = R(f(z)),$$

where $R(f(z))$ is rational in $f(z)$ having constant coefficients, has a non-trivial meromorphic solution regardless of how R is chosen. Yanagihara [69] also considered higher order equations and showed, for instance, that the difference equation

$$\alpha_n f(z+n) + \alpha_{n-1} f(z+n-1) + \cdots + \alpha_1 f(z+1) = R(f(z)), \qquad \alpha_1, \ldots, \alpha_n \in \mathbb{C},$$

has a non-trivial meromorphic solution if the degree p of the numerator $P(f(z))$ of the rational function $R(f(z))$ satisfies $p \geq q+2$, where q is the degree of the denominator $Q(f(z))$ and $P(f(z))$ and $Q(f(z))$ have no common factors.

Ablowitz, Halburd and Herbst [1] suggested that the existence of at least one finite-order meromorphic solutions of a difference equation is a good difference analogue of the Painlevé property. For example, they proved theorems 13 and 14 for the existence of meromorphic solutions of second-order difference equations. Halburd and Korhonen [26] showed that if the equation

$$f(z+1) + f(z-1) = R(z, f(z)), \tag{4.8}$$

where the right-hand side now has meromorphic coefficients, has an admissible meromorphic solution f of finite-order, then either f satisfies a difference Riccati equation or a linear transformation of (4.8) reduces it to one of a short list of difference equations consisting solely of difference Painlevé equations and equations related to them, linear equations and linearizable equations. These results appear to verify that the aforementioned approach by Ablowitz, Halburd and Herbst is a good complex analytic difference analogue of the Painlevé property.

Nakamura and Yanagihara [51] and Yanagihara [70] considered meromorphic solutions of equations of the type

$$f(z+1)^n = P(f(z)), \tag{4.9}$$

where $P(f(z))$ is a polynomial in $f(z)$ with constants as coefficients. Under the assumption that $\deg_f(P(f(z))) = n$, Yanagihara [70] showed that if (4.9) has a transcendental solution f, then (4.9) reduces to the following four equations:

$$\begin{aligned} f(z+1) &= af(z) + b,, && (4.10)\\ f(z+1)^2 &= 1 - f(z)^2,, && (4.11)\\ f(z+1)^2 &= \eta^2(f(z)^2 - 1), && (4.12)\\ f(z+1)^3 &= 1 - f(z)^3,, && (4.13) \end{aligned}$$

where η is the cubic root of 1, $a \neq 0$ and b are constants. In particular, Yanagihara [70] proved that only (4.10) and (4.11) can have finite-order meromorphic solutions. Meromorphic solutions of equation (4.9) under the assumption $\deg_f(P(z, f)) \neq n$ were given in [51].

To provide a difference analogue of Steinmetz' generalization of Malmquist's theorem, it is natural to consider the existence of meromorphic solutions of the equation

$$f(z+1)^n = R(z, f), \tag{4.14}$$

where $R(z, f)$ is rational in both arguments. If equation (4.14) has a transcendental meromorphic solution f such that the hyper-order $\varsigma(f)$ of f satisfies $\varsigma(f) < 1$, then, by (4.14), it follows by Lemma 2 and Theorem 5 that

$$\begin{aligned} \deg_f(R(z, f))T(r, f) &= T(r, R(z, f)) + S(r, f)\\ &= T(r, f(z+1)^n) + S(r, f)\\ &= nT(r, f(z+1)) + S(r, f)\\ &= nT(r, f) + S(r, f), \end{aligned} \tag{4.15}$$

which implies that $\deg_f(R(z, f)) = n$. In this case, it is noted in Paper IV that (4.14) can be reduced to one equation in a list of canonical equations. Details of these equations will be given in Section 5.4

5 Summary of papers

In the following summaries, the notations used in the original papers have been changed to correspond to the previous sections.

5.1 SUMMARY OF PAPER I

We prove an extension of Lemma 2 and use it to extend the estimate (3.1) to the case of meromorphic functions $f(z)$ such that $\log T(r,f) \leq ar/(\log r)^{2+\nu}$, $a, \nu > 0$, for all sufficiently large r. We show the necessity of the exceptional set in (3.1) by proving that this set must be of infinite linear measure for meromorphic functions whose deficiency is dependent on the choice of the origin, such as examples constructed by Gol'dberg [6] and Miles [47], and that there is an infinite sequence of r_n in the set along which $m(r_n, f(z+c)/f(z)) \neq o(T(r_n, f))$ for entire functions constructed by Miles [47]. We also give a discrete version of the Borel-type growth lemma and use it to extend Halburd's result on first-order discrete equations of the Malmquist type.

5.1.1 Borel-type growth lemma extensions

The following Borel-type growth lemma is an extension of Lemma 2.

Lemma 4. *Let $T : [0,\infty) \to (0,\infty)$ be a non-decreasing continuous function and let $s \in (0,\infty)$. If*

$$\limsup_{r\to\infty} \frac{h(r)h(rh(r))\log T(r)}{r} = \zeta, \tag{5.1}$$

where $\zeta \in [0,\infty)$ and $h : [r_0,\infty) \to (0,\infty)$ is an increasing function such that

$$\int_{r_0}^{\infty} \frac{1}{th(t)}\,dt$$

converges, then

$$T(r+s) = T(r) + (\zeta + o(1))\left(\frac{T(r)}{h(r)}\right), \tag{5.2}$$

where r runs to infinity outside of a set E of finite logarithmic measure, i.e., $\int_E dr/r < \infty$.

The idea of the proof of this lemma is the same as that in the proof of Lemma 2 in [30]. Let $\zeta(r)$ be a function of r having a finite limit as $r \to \infty$. If the characteristic function $T(r,f)$ of a meromorphic function $f(z)$ grows in the way that

$$\log T(r,f) \leq \frac{r\zeta(r)}{h(r)h(rh(r))},$$

then, under the assumptions of Lemma 4, $T(r,f)$ satisfies the asymptotic relationship (5.2). However, when $\zeta(r)$ is a function of r which tends to ∞ as $r \to \infty$, the situation will be different. For example, for the function $f(z) = \exp(e^z)$, it follows from [31, p. 7] that

$$\log T(r, f) \sim r - \frac{1}{2}\log r - \frac{1}{2}\log(2\pi^3).$$

We see that in this case $\zeta = \infty$ in (5.1), but $T(r+1, f(z)) = (e + o(1))T(r, f)$. This example shows that Lemma 4 is the best possible estimate when applied to meromorphic functions in the sense that ζ cannot be extended to ∞ in general.

For a given small $\varepsilon > 0$, we may specialize $h(r)$ in Lemma 4 to be $h(r) = (\log r)^{1+\varepsilon}$, where $r \geq r_0 > e$. Thus, when the function $T(r)$ satisfies $\log T(r) \leq ar/(\log r)^{\nu+2}$, where $a, \nu > 0$ and $r \in [r_0, \infty)$, we have

$$\limsup_{r\to\infty} \frac{h(r)h(rh(r))\log T(r)}{r} = 0,$$

and thus by Lemma 4 it follows that for any finite number $s > 0$,

$$T(r+s) = T(r) + o\left(\frac{T(r)}{(\log r)^{1+\varepsilon}}\right), \tag{5.3}$$

for all r outside of a set of finite logarithmic measure. By combining (3.5) and (5.3), we immediately observe that (3.6) also holds for non-constant meromorphic functions f such that $\log T(r, f) \leq ar/(\log r)^{2+\nu}$, $a, \nu > 0$, for all $r \geq r_0 \geq e$. For such meromorphic functions, we prove Lemma 5 below, which is an extension of Theorem 6.

Lemma 5. *Let a and ν be two positive numbers and f a non-constant meromorphic function such that $\log T(r, f) \leq ar/(\log r)^{2+\nu}$ for all sufficiently large r. Then, for a given small $\varepsilon > 0$,*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r, f)}{(\log r)^{\nu-\varepsilon}}\right), \tag{5.4}$$

for all r outside of an exceptional set with finite logarithmic measure.

As in the proof of Theorem 6 in [30], the first step is to show that

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r+|c|, f)}{(\log r)^{\nu-\varepsilon}}\right)$$

holds for all r outside the closed set E, which is defined as

$$E = \left\{r : T\left(r + |c| + \frac{r+|c|}{\xi(T(r+|c|, f))}, f\right) \geq CT(r+|c|, f)\right\}, \tag{5.5}$$

where $\xi(x) = (\log x)(\log\log x)^{1+\varepsilon}$. The logarithmic measure of this set satisfies

$$\int_{E\cap[r_0,R]} \frac{dr}{r} \leq \frac{1}{\log C}\int_{x_0}^{T(R,f)} \frac{dx}{x\xi(x)} + O(1), \tag{5.6}$$

where $R < \infty$. It follows that according to Borel's lemma [11] the logarithmic measure of E above is finite.

5.1.2 Necessity of the exceptional set

We begin from a classic result by Valiron [64] concerning the dependence on the choice of origin of the deficiency of a meromorphic function f. We first prove the following Proposition 1, which is the counterpart of Valiron's result [64]: If the characteristic function $T(r,f)$ of a meromorphic function f satisfies the condition

$$\lim_{r\to\infty}\frac{T(r+1,f)}{T(r,f)}=1, \tag{5.7}$$

then the deficiency of f is independent of the choice of the origin, that is, $\delta(0,f(z))=\delta(0,f(z+c))$ for any finite nonzero constant c. Moreover, if f has finite order σ and lower order μ, then the condition (5.7) can also be replaced by $\sigma-\mu<1$. Chiang and Luo [16, p. 455] pointed out that the estimate (3.7) together with [12, Theorem 2.1] implies the finite order case of Valiron's result since the error term $O(r^{\sigma-1+\varepsilon})$ in (3.7) is small compared with $T(r,f)$ as $r\to\infty$ without any exceptional set.

Proposition 1. *Let f be a non-constant meromorphic function. If $\delta(0,f(z))>\delta(0,f(z+c))$ for some $c\neq 0$, then there exists a constant $1<C<\infty$ and a set E_0 with infinite linear measure such that*

$$T(r+|c|,f)\geq CT(r,f),$$

for all $r\in E_0$. Moreover, if f is entire, then there is an infinite sequence of r_n such that as $r_n\to\infty$,

$$m\left(r_n,\frac{f(z+c)}{f(z)}\right)\neq o(T(r_n,f)).$$

More specifically, by the definition of $\delta(0,f(z+c))$, there is an infinite sequence of r_n such that $m(r_n,1/f(z+c))\leq(\delta(0,f(z+c))+\varepsilon)T(r_n,f(z+c))$, where $\varepsilon>0$ is a given constant and r_n is sufficiently large, and along this sequence of r_n, we have

$$m\left(r_n,\frac{f(z+c)}{f(z)}\right)\geq\frac{\delta(0,f(z))-\delta(0,f(z+c))-2\varepsilon}{1+\delta(0,f(z+c))+\varepsilon}\cdot T(r_n,f(z)). \tag{5.8}$$

Meromorphic functions having the property $\delta(0,f(z))\neq\delta(0,f(z+c))$ for some nonzero constant c do exist, see Miles [47]. In the finite order case, we recall the following two examples. The first one is based on Gol'dberg [6], who constructed a meromorphic function with order 1 such that

$$\begin{aligned}&\delta(0,f(z))=1, \quad \text{and}\\ &\delta(0,f(z+c))=0, \quad \text{for some } c\neq 0.\end{aligned}$$

This example also shows that the finite order case of Valiron's result above is sharp. The second one is based on Miles [47], who proved there exists an entire function f of order $3/2<\sigma(f)<\infty$ such that

$$\begin{aligned}&\delta(0,f(z))=0, \quad \text{and}\\ &\delta(0,f(z+c))\geq\rho>0, \quad \text{for all } c\neq 0,\end{aligned}$$

for some $\rho<1$ independent of c. A simple counterexample for Proposition 1 is the function $f(z)=e^z$ with the characteristic $T(r,f)=r/\pi$. This function satisfies $\delta(0,f(z))=\delta(0,f(z+c))=1$ for any nonzero constant c while $T(r+|c|,f)=T(r,f)+O(1)$ and $m(r,f(z+c)/f(z))=O(1)$.

Now we use this Proposition 1 to study the necessity of the exceptional set in the lemma on the difference quotients, i.e., in Theorem 6. Let f be a non-constant meromorphic function of hyper-order $\varsigma < 1$ such that $\delta(0, f(z)) > \delta(0, f(z+c))$ for some nonzero constant c. We need to choose $\xi(x)$ in Lemma 5 to be $\xi(x) = (\log x)^{1+\varepsilon}$, where $\varepsilon > 0$ and satisfies $(\varsigma + \varepsilon)(1+\varepsilon) < 1$, to obtain the estimate (3.1) as in the proof of Theorem 6 and, consequently, the set E in (5.5) is redefined to be

$$E = \left\{ r : T\left(r + |c| + \frac{r + |c|}{(\log T(r+|c|, f))^{1+\varepsilon}}, f \right) \geq CT(r+|c|, f) \right\}. \tag{5.9}$$

Note that $T(r, f)$ satisfies the asymptotic relation (3.4) outside of the set of finite logarithmic measure. Proposition 1 implies that this set is of infinite linear measure. However, from (3.2) we see that the exceptional set in the estimate (3.1) is independent of that in (3.4) whenever the characteristic function $T(r, f)$ satisfies

$$\limsup_{r\to\infty} \frac{T(r+|c|, f)}{T(r, f)} < \infty.$$

This possibility cannot be excluded automatically when $\delta(0, f(z+c)) < 1$, as shown in the proof of Proposition 1 in Paper I. Thus we need to consider the set (5.9). Firstly, it follows from Proposition 1 that for some finite constant $C > 1$ the set

$$E_2 = \{r : T(r + 2|c|, f) \geq CT(r+|c|, f)\}$$

is of infinite linear measure. From the definition of E in (5.9), it is evident that there is a sufficiently large r_0 such that $E_2 \cap [r_0, \infty) \subseteq E \cap [r_0, \infty)$. Thus the exceptional set associated with the error term in the estimate (3.1) is also of infinite linear measure. Secondly, if f is entire, we know from Proposition 1 that there is an infinite sequence of r_n such that $m(r_n, f(z+c)/f(z)) \neq o(T(r_n, f))$ as $r_n \to \infty$. It follows from (3.1) that the infinite sequence of r_n satisfying (5.8) must be in the exceptional set associated with the error term in (3.1). In conclusion, for entire functions of hyper-order less than 1 whose deficiency is dependent on the choice of origin, the estimate (3.1) is not applicable in the exceptional set with infinite linear measure. This shows the necessity of the exceptional set in the estimate (3.1).

5.1.3 Discrete Borel-type growth lemma extensions

By replacing the continuous variable r in Lemma 4 with a sequence of positive numbers, we have the following discrete analogue of Lemma 4.

Lemma 6. *Let $(T_n)_{n\geq n_0}$ $(n_0 > 0)$ be a non-decreasing sequence of positive numbers and let s be a fixed positive integer. If*

$$\limsup_{n\to\infty} \frac{h(n)h(nh(n))\log T_n}{n} = \zeta, \tag{5.10}$$

where $\zeta \in [0, \infty)$ and $h(n)$ is an increasing sequence of positive numbers such that

$$\sum_{n=n_0}^{\infty} \frac{1}{nh(n)} < +\infty,$$

then

$$T_{n+s} = T_n + (\zeta + o(1))\left(\frac{T_n}{h(n)}\right), \tag{5.11}$$

where n runs to infinity outside of a set E of finite discrete logarithmic measure, that is, $\sum_{n\in E} 1/n < \infty$.

Analogous to the continuous case, we can observe that for a function $T(n)$ of n such that $\log T(n) \leq an/(\log n)^{2+\nu}$, where $a, \nu > 0$ and $n \in [n_0, \infty)$,

$$T(n+s) = T(n) + o\left(\frac{T(n)}{(\log n)^{1+\varepsilon}}\right).$$

Furthermore, from the characteristic function $T(r, f)$ of the function $f(z) = \exp(e^z)$ we know that the constant ζ in Lemma 6 cannot be extended to ∞ since we may choose an infinite sequence of r_n such that $r_n = n$.

As an application of Lemma 6, we provide an improvement of Halburd's result on the first order discrete equations

$$y_{n+1} = R(n, y_n) = \frac{P(n, y_n)}{Q(n, y_n)}, \tag{5.12}$$

where $P(n, y_n)$ and $Q(n, y_n)$ are coprime polynomials in y_n having rational coefficients in $\mathbb{Q}[n]$. Before stating the result, we need one additional definition. Let k be a number field, and let $(y_n)_{n\in\mathbb{N}} \subset k$ be a solution of (5.12), where the coefficients are in $k[n]$. For $x \in k$ we denote by $H(x)$ the *height* and by $h(x) = \log H(x)$ *the logarithmic height* of x. We say that $(y_n)_{n\in\mathbb{N}}$ is *admissible* if the logarithmic heights of all coefficients of (5.12) are of the growth $o(h(y_n))$ as $n \to \infty$ outside of an exceptional set of finite discrete logarithmic measure $\sum_{n\in E} 1/n < \infty$. This definition is an exact Diophantine analogue of the notion of admissible meromorphic solution of a difference equation in the spirit of Vojta's dictionary [65].

Theorem 19. *Let k be a number field, and let $(y_n)_{n\in\mathbb{N}} \subset k$ be an admissible solution of* (5.12), *where the coefficients are in $k[n]$. If*

$$\limsup_{n\to\infty} \frac{\log \sum_{k=1}^{n} h(y_k)}{n/(\log n)^{2+\nu}} = 0$$

for any $\nu > 0$, then $\deg_{y_0}(R) = 1$.

5.2 SUMMARY OF PAPER II

We first present a result on the growth of meromorphic solutions of the following nonlinear difference equation

$$f(z)^n + P_{n-1}(z, f(z)) = 0, \tag{5.13}$$

where $n \geq 2$ and $P_{n-1}(z, f(z))$ is a difference polynomial in $f(z)$ of degree at most $n-1$ with meromorphic coefficients. Then we study the value distribution of meromorphic solutions of infinite order of the following difference equation

$$f(z)^n + q(z)f(z+1) = c \sin bz, \tag{5.14}$$

where q is a non-constant polynomial, b, c are nonzero constants and $n \geq 2$ is an integer. We also present counterexamples to show that one of the conjectures proposed by Yang and Laine [71] is not true in general.

5.2.1 Growth of meromorphic solutions of nonlinear difference equations

In Paper II, we prove the following Theorem 20 for the admissible case of (5.13).

Theorem 20. *All admissible meromorphic solutions $f(z)$ of* (5.13) *have hyper-order at least 1.*

In the original version of Theorem 20, we only use the first version of Theorem 10 for finite-order meromorphic functions and obtain that $f(z)$ has infinite order. However, since the lemma on the difference quotients has been extended to the case of meromorphic solutions of hyper-order less than 1, it is natural to renew our result in the present form stated above.

For the special case that all the coefficients of (5.13) are rational functions, we can give a more precise lower bound on the growth of the solution $f(z)$. We prove

Theorem 21. *Suppose that $f(z)$ is a transcendental meromorphic solution of* (5.13), *and that all the coefficients of $P_{n-1}(z, f)$ are rational, and that all the shifts of $f(z)$ in $P_{n-1}(z, f)$ are $f(z+c_1), \cdots, f(z+c_k)$. Denote $C = \max\{|c_1|, \cdots, |c_k|\}$ and $m = \frac{n}{n-1}$. Then*

1 *If $f(z)$ is entire or has finitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that*

$$\log M(r, f) \geq K m^{r/C}$$

holds for all $r \geq r_0$;

2 *If $f(z)$ has infinitely many poles, then there exist constants $K > 0$ and $r_0 > 0$ such that*

$$n(r, f) \geq K m^{r/C}$$

holds for all $r \geq r_0$.

The method for the proof of Theorem 20 is originally from [24] and we have quoted a refined version in the proof of [39, Proposition 5.4], while the method to prove Theorem 21 is an adaptation of the proof of [33, Theorem 10].

5.2.2 Counterexamples for Laine and Yang's conjecture

Recall the following two conjectures concerning the existence or non-existence of meromorphic solutions of nonlinear difference equations from [71].

Conjecture 1 (see [71]). *There exists no entire function of infinite order that satisfies a difference equation of the type*

$$f(z)^n + q(z)f(z+1) = c\sin bz, \tag{5.15}$$

where q is a non-constant polynomial, b, c are nonzero constants and $n \geq 2$ is an integer.

Conjecture 2 (see [71]). *Let f be an entire function of infinite order and $n \geq 2$ an integer. Then a differential-difference polynomial of the form $f^n + P_{n-1}(z, f)$ cannot be a non-constant entire function of finite order. Here $P_{n-1}(z, f)$ is a differential-difference polynomial of total degree at most $n-1$ in f, its derivatives and its shifts, with entire functions of finite order as coefficients. Moreover, we assume that all terms of $P_{n-1}(z, f)$ have total degree ≥ 1.*

Until now there is still no example to overturn Conjecture 1. We study the value distribution of meromorphic solutions of (5.15) for the case where the hyper-order of $f(z)$ is less than ∞ and prove the following theorem.

Theorem 22. *Suppose that $f(z)$ is an entire solution of infinite order satisfying* (5.15). *If $\varsigma(f) < \infty$, then for any entire function $d(z)$ of finite order, we have $\lambda(f-d) = \infty$.*

Li [41] has proved that Conjecture 2 is true for meromorphic functions of hyper-order less than 1 by using Theorem 6. We present the following examples to show that it is not true for meromorphic functions with hyper-order greater than or equal to 1 in general: From [54, Theorem 1] follows that for any $1 \leq \sigma < \infty$ there exists a periodic function $\Pi(z)$ with period 1 such that $\sigma(\Pi(z)) = \sigma$. Let $g(z) = \Pi(z)e^{z\log 2}$ and $f(z) = e^{g(z)} + (\log 2)g(z)$. Then $f(z)$ is of infinite order and satisfies the following equation

$$f(z)^2 - f(z+1) - 2f'(z) = (\log 2)^2 g(z)^2 - 2(\log 2)g(z) - 2(\log 2)^2 g(z) \not\equiv 0.$$

It is evident that $\varsigma(f(z)) = \sigma(g(z)) = \sigma \geq 1$. Thus Conjecture 2 does not hold in general when $f(z)$ is of hyper-order $1 \leq \varsigma(f) < \infty$.

5.3 SUMMARY OF PAPER III

We give a positive answer to the question: does the conclusion of Theorem 17 still hold for meromorphic functions of hyper-order strictly less than 1 when replacing the shift $f(z+\eta)$ with the n^{th} order difference $\Delta_\eta^n f(z)$? We prove the following Theorem 23, which can be viewed as a difference analogue of Theorem 18.

Theorem 23. *Let $f(z)$ be a transcendental meromorphic function of hyper-order strictly less than 1 such that $\Delta^n f(z) \not\equiv 0$. If $f(z)$ and $\Delta^n f(z)$ share three distinct periodic functions $a, b, c \in \hat{\mathcal{S}}(f)$ with period 1 CM, then $\Delta^n f(z) \equiv f(z)$.*

Let $f(z)$ be a meromorphic solution of the linear difference equation: $\Delta^n f(z) - 1 = \xi(f(z)-1)$, where $\xi \neq 0, 1$ is a constant. Then $\Delta^n f(z)$ and $f(z)$ share the values $1, \infty$ CM, but $\Delta^n f(z) \not\equiv f(z)$. This counterexample shows that the 3 CM condition in Theorem 23 cannot be reduced to any 2 CM condition.

In Theorem 23, we have assumed $\Delta^n f(z) \not\equiv 0$ since $\Delta^n f(z) = 0$ is an n^{th} order linear difference equation in $f(z)$ with constant coefficients and can be solved explicitly. Consider the meromorphic solution $f(z)$ of the linear difference equation: $\Delta^n f(z) = f(z)$. The distinct roots of $\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\lambda^k = 1$ are $\lambda_k = 1 + e^{\frac{2k\pi i}{n}}$, $k = 0, \ldots, n-1$, thus $f(z)$ can be expressed as $f(z) = \sum_{k=0}^{n-1} \pi_k(z)\lambda_k^z$, $k = 0, \ldots, n-1$, where $\pi_k(z)$ are arbitrary 1-periodic functions. We conjecture that the conclusion of Theorem 23 is still true when the assumption $\varsigma(f) < 1$ is dropped.

The proof of Theorem 23 is divided into two cases: $c \equiv \infty$ or $c \not\equiv \infty$. One of the key lemmas applied in the proof of Theorem 23 is Lemma 3: Let $\eta \in \mathbb{C}$, and $g_0, \ldots, g_n$ be entire functions such that $\varsigma(g_i) < 1$, $i = 0, \ldots, n$ and such that all zeros of $g_0, \ldots, g_n$ are forward invariant with respect to the translation $\tau(z) = z + \eta$. If $g_i/g_j \notin \mathcal{P}_\eta^1$ for all $i, j \in \{0, \ldots, n\}$ such that $i \neq j$, then $g_0, \ldots, g_n$ are linearly independent over $\mathcal{P}_\eta^1$. Under the assumption that $\Delta^n f(z) \not\equiv f(z)$, we apply this lemma in the case that $abc \not\equiv \infty$, where a suitable transformation gives two functions $g_1(z)$ and $g_2(z)$ of the form

$$g_1(z) = \frac{e^\alpha(1-e^\beta)}{e^\alpha - e^\beta}, \quad g_2(z) = \frac{1-e^\beta}{e^\alpha - e^\beta},$$

and $g_1(z)$, $g_2(z)$ share $0, 1, \infty$. After some calculations we obtain

$$ade^{h_1} + (b-ad)e^{h_2} + \sum_{j=3}^{m} c_j e^{h_j} = 0, \tag{5.16}$$

where $c_3, \ldots, c_m$, $(m \in \mathbb{N}^+)$ are nonzero periodic functions of period 1 and

$$h_1 = \sum_{i=0}^{n}(\alpha_i + \beta_i) + \alpha, \quad h_2 = \sum_{i=0}^{n}(\alpha_i + \beta_i) + \beta, \quad h_j = \sum_{i=0}^{n}(l_i\alpha_i + s_i\beta_i),$$

and l_i, s_i are integers such that $0 \leq l_i, s_i \leq 2$. Moreover, the combined number of α_i and β_i corresponding to h_j, $j \in \{3, \ldots, m\}$ is at least 1 and at most $2n+2$. Then, by Lemma 3 we conclude that e^α and e^β satisfy one of the three relations: (1) $T(r, e^\beta) = S(r, f)$; (2) $T(r, e^\alpha) = S(r, f)$; or (3) $e^\beta = He^{t\alpha}$ for some meromorphic function H such that $T(r, H) = S(r, f)$, but none of these cases can occur implying that $\Delta^n f(z) \equiv f(z)$.

5.4 SUMMARY OF PAPER IV

If the hyper-order of $f(z)$ is less than one, then by Lemma 2 it follows that shifting a difference equation does not affect the admissibility of the solutions. If the hyper-order of $f(z)$ is at least one, then the possibility of losing admissibility when shifting a difference equation cannot be automatically ruled out. We define the following set for small functions of a meromorphic function $f(z)$,

$$\mathcal{S}'(f) = \{c(z) \in \mathcal{M} : T(r, c(z+n)) = S(r, f(z+n)), n \in \mathbb{N}\},$$

and in the following we say that a meromorphic solution $f(z)$ of a difference equation is admissible when all coefficients of the equation are in $\mathcal{S}'(f)$. For simplicity, below we will use the suppressed notations: $f = f(z)$, $\overline{f} = f(z+1)$ for a meromorphic, or algebroid, function $f(z)$. Assume that all the coefficients of

$$\overline{f}^n = R(z, f), \tag{5.17}$$

are in $\mathcal{S}'(f)$ and that $\deg_f(R(z,f)) = n$. We obtain the following theorem, which is a further extension of the results on (4.9) of Yanagihara [70].

Theorem 24. *Let $n \in \mathbb{N}$. If the difference equation* (5.17) *with* $\deg_f(R(z,f)) = n$ *has an admissible meromorphic solution, then f either satisfies a difference linear or Riccati equation:*

$$\begin{aligned} \overline{f} &= a_1 f + a_2, && (5.18)\\ \overline{f} &= \frac{b_1 f + b_2}{f + b_3}, && (5.19) \end{aligned}$$

where a_i, b_j are small meromorphic functions; or, by a transformation $f \to \alpha f$ or $f \to 1/(\alpha f)$ with a small algebroid function α of degree at most 3, (5.17) *reduces to one of the following equations:*

$$\begin{aligned} \overline{f}^2 &= 1 - f^2, && (5.20)\\ \overline{f}^2 &= \delta_1(f^2 - 1), && (5.21)\\ \overline{f}^2 &= 1 - \left(\frac{\delta_2 f - 1}{f - \delta_2}\right)^2, && (5.22)\\ \overline{f}^2 &= \delta_3(1 - f^{-2}), && (5.23)\\ \overline{f}^2 &= 1 - \left(\frac{f+3}{f-1}\right)^2, && (5.24)\\ \overline{f}^2 &= \frac{f^2 - \kappa_1^2}{f^2 - 1}, && (5.25)\\ \overline{f}^2 &= \frac{\overline{\kappa}_2^2 f^2 - 1}{f^2 - 1}, && (5.26)\\ \overline{f}^2 &= \theta \frac{f^2 - \kappa_3 f + 1}{f^2 + \kappa_3 f + 1}, && (5.27)\\ \overline{f}^3 &= 1 - f^3, && (5.28)\\ \overline{f}^3 &= 1 - f^{-3}, && (5.29) \end{aligned}$$

where $\theta = \pm 1$, $\delta_2 \not\equiv \pm 1$ *is a small algebroid function of degree at most 2 and* δ_1, δ_3, κ_1^2, κ_2^2, κ_3^2 *are all small meromorphic functions satisfying* $\overline{\delta}_1(\delta_1+1)+1=0$, $\overline{\delta}_3\delta_3 = \overline{\delta}_3+\delta_3$, $\overline{\kappa}_1^2 = \kappa_1^2$, $\overline{\kappa}_2^2\kappa_2^2 = 1$ *and* $\overline{\kappa}_3^2(\kappa_3^2-4) = 2(1-\theta)\kappa_3^2 - 8(1+\theta)$.

Assume that in Theorem 24 all the coefficients of (5.17) are rational functions. Then the coefficients $\delta_1, \delta_3, \kappa_1^2, \kappa_2^2, \kappa_3^2$ are all constants and the algebraic case of α can only occur when obtaining equation (5.22). In this case, solutions to (5.28) and (5.29) can be characterized by Weierstrass elliptic functions, solutions to equations (5.21), (5.23), (5.25)–(5.27) can be characterized by Jacobi elliptic functions and equations (5.20), (5.22) and (5.24) can be explicitly solved in terms of functions which are solutions of certain difference Riccati equations.

Equations (5.20), (5.22), (5.24), (5.28) and (5.29) are so-called *Fermat* difference equations. In general, a Fermat equation is a function analogue of the Fermat diophantine equation $x^n + y^n = 1$, i.e., $h(z)^n + g(z)^n = 1$, where $n \geq 2$ is an integer. Meromorphic solutions to Fermat equations have been clearly characterized, see [2,20,21], for example. In particular, when $n = 3$, all meromorphic solutions can be represented as: $h = H(\varphi)$, $g = \eta G(\varphi) = \eta H(-\varphi) = H(-\eta^2\varphi)$, where $\varphi = \varphi(z)$ is an entire function and η is the cubic root of 1, and

$$H(z) = \frac{1+\wp'(z)/\sqrt{3}}{2\wp(z)}, \quad G(z) = \frac{1-\wp'(z)/\sqrt{3}}{2\wp(z)}, \tag{5.30}$$

is a pair of solutions of the Fermat equation with $\wp(z)$ being the particular Weierstrass elliptic function satisfying $\wp'(z)^2 = 4\wp(z)^3 - 1$. For equation (5.28) the solution then takes the form: $f = H(\varphi)$, $\overline{f} = \eta G(\varphi) = H(-\eta^2\varphi)$. It follows that $\overline{\varphi} = -\eta^2\varphi + \omega$, where ω is a period of $\wp(z)$, which implies that φ is a transcendental entire function of order at least 1. From (5.28) we have $T(r,\overline{f}) = T(r,f) + O(1)$. Moreover, we have $\eta f + \overline{f} = \eta/\wp(\varphi)$, which yields $T(r,\wp(\varphi)) \leq 2T(r,f) + O(1)$. By taking the derivatives on both sides of $\eta f + \overline{f} = \eta/\wp(\varphi)$ and combining it with the resulting equation, we get

$$\varphi' = -\frac{(\eta f + \overline{f})'}{\eta f + \overline{f}} \cdot \frac{\wp(\varphi)}{\wp'(\varphi)}. \tag{5.31}$$

For the elliptic function $\wp(z)$, we have $N(r,\wp(\varphi)) = T(r,\wp(\varphi)) + O(\log rT(r,\wp(\varphi)))$ and it follows that $m(r,\wp(\varphi)) = O(\log rT(r,\wp(\varphi))) = O(\log rT(r,f))$ and, similarly, $m(r,1/\wp'(\varphi)) = O(\log rT(r,f))$. Thus, by taking the proximity functions on both sides of (5.31), we get

$$m(r,\varphi') = O(\log rT(r,f)) + O(1).$$

Then, $T(r,\varphi') = m(r,\varphi') = O(\log rT(r,f)) + O(1)$, which yields that $\sigma_2(f) \geq 1$ together with [38, Lemma 1.1.1] and so equation (5.28) cannot have meromorphic solutions of hyper-order less than 1.

The autonomous versions of (5.21), (5.23), (5.26) and (5.27) cannot admit meromorphic solutions of hyper-order < 1 either. Solutions to them can be characterized by the Jacobi elliptic function $\mathrm{sn}(z,k)$ with the elliptic modulus $k \in (0,1)$, which satisfies the differential equation $\mathrm{sn}'(z)^2 = (1-\mathrm{sn}(z)^2)(1-k^2\mathrm{sn}(z)^2)$. For equation (5.23), $\delta_3 = 2$ and we see that f^2-1 is written as $f^2 - 1 = -(i\overline{f}f/\sqrt{2})^2 = -f_0^2$. Then we have $T(r,f) = T(r,f_0) + O(1)$ and by substituting $f^2 = -f_0^2 + 1$ into (5.23) we get the autonomous case of equation (5.26). For equation (5.27), if we let $w = f + 1/f$, then w satisfies $T(r, w) = 2T(r, f) + O(1)$ by Valiron's identity [63] (see also [38]) and it follows that

$$\overline{f}^2 = \theta\frac{w - \kappa_3}{w + \kappa_3}, \quad \frac{1}{\overline{f}^2} = \theta\frac{w + \kappa_3}{w - \kappa_3}.$$

The above two equations yield

$$\overline{w}^2 = \frac{2(\theta + 1)w^2 + 2(\theta - 1)\kappa_3^2}{w^2 - \kappa_3^2}.$$

If $\theta = 1$, then we have $\overline{\kappa}_3^2(\kappa_3^2 - 4) = -16$ and by performing a transformation $w \to \kappa_3/w$ we get $\overline{w}^2 = d_1(w^2 - 1)$, which is equation (5.21) since $d_1 = -\overline{\kappa}_3^2/4$ satisfies $\overline{d}_1(d_1 + 1) + 1 = 0$; if $\theta = -1$, then we have $\overline{\kappa}_3^2(\kappa_3^2 - 4) = 4\kappa_3^2$ and by performing a transformation $w \to \kappa_3/w$ we get $\overline{w}^2 = d_2(1 - w^{-2})$, which is the equation (5.23) since $d_2 = \overline{\kappa}_3^2/4$ satisfies $\overline{d}_2 d_2 = \overline{d}_2 + d_2$. Thus we only need to consider solutions of equations (5.21) and (5.26). Consider equation (5.21) as an example. Now δ_1 is the cubic root of 1 and we see that f is twofold ramified over ± 1. Denote $\delta_1 = \eta^2$, where η is the cubic root of unity and satisfies $\eta^2(\eta^2 + 1) + 1 = \eta^2 + \eta + 1 = 0$. Let z_0 be such that $f(z_0) = \pm i\eta$, then by (5.21) we have $f(z_0 + 1) = \pm 1$, and so f is also twofold ramified at $\pm i\eta$-points. Let $\varrho_1 = \exp(7i\pi/12)$ and denote $\tau_1 = (1 - \varrho_1)/(1 + \varrho_1)$. Then

$$g_1 = \tau_1 \frac{f + \varrho_1}{f - \varrho_1}$$

is twofold ramified over each of $\pm 1, \pm\tau_1^2$, where $-\tau_1^2 = \tan(7\pi/24)^2 > 1$. Let $\mathrm{sn}(\phi) = \mathrm{sn}(\phi, -1/\tau_1^2)$ and ϕ_0 be such that $\mathrm{sn}(\phi_0) \neq \pm 1, \pm\tau_1^2$. Letting z_0 be such that $g_1(z_0) = \mathrm{sn}(\phi_0)$, it follows that there is a neighborhood U of z_0 such that $\phi_1(z) = \mathrm{sn}^{-1}(g_1(z))$ is defined and holomorphic in U. By following the reasoning in the proof of [70, Lemma 4.1], we know that $\phi_1(z)$ can be continued analytically throughout the complex plane to an entire function. Therefore, g_1 is written as $g_1(z) = \mathrm{sn}(\phi_1(z))$, where $\mathrm{sn}(\phi_1) = \mathrm{sn}(\phi_1, -1/\tau_1^2)$ satisfies the differential equation $\mathrm{sn}'(\phi_1)^2 = (1 - \mathrm{sn}(\phi_1)^2)(1 - \mathrm{sn}(\phi_1)^2/\tau_1^4)$. By taking the derivative of g_1 and combining it with the resulting equation, we get

$$\phi_1'^2 = \frac{\mathrm{sn}(\phi_1)^2}{\mathrm{sn}'(\phi_1)^2} \cdot \left(\frac{g_1'}{g_1}\right)^2 = \frac{g_1^2}{(1 - g_1^2)(1 - g_1^2/\tau_1^4)} \cdot \left(\frac{g_1'}{g_1}\right)^2. \tag{5.32}$$

Since g_1 is an elliptic function, we have $m(r, g_1) = O(\log rT(r, f))$ and $m(r, 1/(g_1 - \varrho)) = O(\log rT(r, f))$, $\varrho = \pm 1, \pm\tau_1^2$. Note that $T(r, f) = T(r, g_1) + O(1)$. Taking the proximity function on both sides of (5.32) gives $T(r, \phi_1'^2) = m(r, \phi_1'^2) = O(\log rT(r, f)) + O(1)$, which together with [38, Lemma 1.1.1] demonstrates that ϕ_1' has order of growth strictly less than 1 when $\sigma_2(f) < 1$. By iterating (5.21) we obtain $f(z + 3)^2 = f(z)^2$ and so $f(z + 6) = f(z)$ and it follows that $\mathrm{sn}(\phi_1(z + 6)) = \mathrm{sn}(\phi_1(z))$ giving $\phi_1(z + 6) = \phi_1(z) + K_1$ with a period K_1, which is possible only when ϕ_1 is a polynomial of degree 1 since $\phi_1'(z + 6) = \phi_1'(z)$. This implies that f is of finite order. But according to [70, Lemma 9.1], (5.21) cannot admit any meromorphic solutions of finite order when δ_1 is a constant. Therefore, (5.21) cannot admit any meromorphic solutions of hyper-order strictly less than 1 when δ_1 is a constant. Using similar arguments as above we can also obtain that (5.26) cannot admit any meromorphic solutions of hyper-order strictly less than 1 when κ_2 is a constant.

If equation (5.17) has an admissible meromorphic solution f such that the hyper-order $\varsigma(f)$ of f satisfies $\varsigma(f) < 1$, then $\deg_f(R(z,f)) = n$. Thus we have the following corollary which is a natural difference analogue of Steinmetz's generalization [62] of Malmquist's 1913 result on differential equations.

Corollary 1. *Let $n \in \mathbb{N}$. If the difference equation* (5.17) *with rational coefficients has a transcendental meromorphic solution of hyper-order strictly less than 1, then either f satisfies* (5.18) *or* (5.19) *with rational coefficients or, by a transformation $f \to \alpha f$ or $f \to 1/(\alpha f)$ with an algebraic function α of degree at most 2,* (5.17) *reduces to one of the following equations:*

$$\begin{aligned}
\overline{f}^2 &= 1 - f^2, && (5.33)\\
\overline{f}^2 &= 1 - \left(\frac{\delta f - 1}{f - \delta}\right)^2, && (5.34)\\
\overline{f}^2 &= 1 - \left(\frac{f + 3}{f - 1}\right)^2, && (5.35)\\
\overline{f}^2 &= \frac{f^2 - \kappa^2}{f^2 - 1}, && (5.36)\\
\overline{f}^3 &= 1 - f^{-3}, && (5.37)
\end{aligned}$$

where $\delta \not\equiv \pm 1$ is an algebraic function of degree 2 at most and $\kappa^2 \neq 0, 1$ is a constant.

Equations (5.33)-(5.37) indeed have meromorphic solutions of finite order.

Let us first look at the second-degree Fermat difference equations (5.33), (5.34) and (5.35) first. For equation (5.33), we know from [70, Theorem 2] that the solution f is represented as $f = (\beta + \beta^{-1})/2$, where β satisfies $\overline{\beta} = i\beta^{\pm 1}$. For equation (5.34), if we put $f = (\gamma + \gamma^{-1})/2$, we then have

$$\frac{1}{4}\left(\overline{\gamma} - \frac{1}{\overline{\gamma}}\right)^2 = -\left(\frac{\delta f - 1}{f - \delta}\right)^2.$$

It follows that

$$\overline{\gamma}^2 - 2i\frac{\delta\gamma^2 - 2\gamma + \delta}{\gamma^2 - 2\delta\gamma + 1}\overline{\gamma} - 1 = 0.$$

Solving the above equation, we get the difference Riccati equation

$$\overline{\gamma} = \left\{-\theta\frac{(i\delta - \sqrt{1-\delta^2})\gamma + i}{\gamma - \delta + i\sqrt{1-\delta^2}}\right\}^{\theta}, \quad \theta = \pm 1.$$

For equation (5.35), we have

$$\overline{f}^2 = -\frac{8(f+1)}{(f-1)^2}.$$

If we put

$$\sqrt{-8}u = \frac{\overline{f}(f-1)}{f+1}, \quad v = \frac{1}{f+1},$$

then we have

$$u^2 = v, \quad f = \frac{1}{v} - 1, \quad \overline{f} = \frac{\sqrt{-8}u}{1 - 2v},$$

and further

$$(\sqrt{2}\overline{u})^2 = 2\overline{v} = \frac{2}{\overline{f}+1} = \frac{2(2u^2-1)}{2u^2-\sqrt{-8}u-1} = 1 - \left(\frac{\sqrt{2}iu-1}{\sqrt{2}u-i}\right)^2.$$

Putting $\sqrt{2}u = (\lambda + \lambda^{-1})/2$, then we have

$$f = \frac{1-u^2}{u^2} = \frac{8\lambda^2 - (\lambda^2+1)^2}{(\lambda^2+1)^2}, \quad \overline{\lambda} = \left\{-\theta\frac{-(1+\sqrt{2})\lambda + i}{\lambda - i + i\sqrt{2}}\right\}^{\theta}, \quad \theta = \pm 1.$$

Since the autonomous version of the difference Riccati equation is solvable explicitly in terms of exponential functions, so are equations (5.33), (5.34) and (5.35) when δ is a constant.

Equation (5.36) can be rewritten as $\overline{f}^2 f^2 - (\overline{f}^2 + f^2) + \kappa^2 = 0$, which is a special case of symmetric QRT map [57, 58]. By performing a suitable Möbius transformation [5, 59], for example, $f \to a(f+1)/(f-1)$, where a ($|a| > 1$) is a constant satisfying $2a^4 - 2a^2 - 1 = 0$ and $\kappa^2 = a^4$, we get

$$\overline{f}^2 f^2 + \overline{f}^2 + f^2 + 4(1+4a^2)\overline{f}f + 1 = 0,$$

which is solvable in terms of Jacobi elliptic functions with finite order of growth [27].

For the third degree Fermat difference equation (5.37), the solution f satisfies $f^{-1} = H(\varphi), \overline{f} = \eta G(\varphi)$, where H, G are defined in (5.30) and φ is an entire function. Choose $\eta = 1$. It follows that

$$\frac{1+\wp'(\overline{\varphi})/\sqrt{3}}{2\wp(\overline{\varphi})} \cdot \frac{1-\wp'(\varphi)/\sqrt{3}}{2\wp(\varphi)} = 1.$$

Using the addition law of the Weierstrass elliptic function together with the relation $\wp'^2 = 4\wp^3 - 1$, it can be shown that this equation is solved by a polynomial of degree 1 satisfying $\overline{\varphi} = \varphi + a$, where a is a constant such that $\wp'(a) = -\sqrt{3}$ and $\wp(a) = 1$. It follows that the order of growth of f is 2.

In the proof of Theorem 24, we have mainly used the ramification theorem, i.e., Theorem 4 to rule out cases of (5.17) without meromorphic solutions. We first find some restrictions on the roots and degrees of the numerator and denominator of $R(z, f)$, which allows us to only consider two cases of equation (5.17) where $p = n, q = 0$ or $p = q = n$ after a possible bilinear transformation to f. The results obtained in the case $p = n, q = 0$ are the admissible counterpart of the results on (4.9) by Yanagihara [70].

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