PUBLICATIONS OF
THE UNIVERSITY OF EASTERN FINLAND

Dissertations in Forestry and Natural Sciences

UNIVERSITY OF EASTERN FINLAND

KIAN SIERRA MCGETTIGAN

CLASSICAL OPERATORS ON WEIGHTED BERGMAN AND MIXED NORM SPACES

PUBLICATIONS OF THE UNIVERSITY OF EASTERN FINLAND
DISSERTATIONS IN FORESTRY AND NATURAL SCIENCES

N:o 300

Kian Sierra McGettigan

CLASSICAL OPERATORS ON WEIGHTED BERGMAN AND MIXED NORM SPACES

ACADEMIC DISSERTATION

To be presented by the permission of the Faculty of Science and Forestry for public examination in the Auditorium M301 in Faculty of Science and Forestry at the University of Eastern Finland, Joensuu, on March, 16, 2018, at 12 o'clock.

University of Eastern Finland
Department of Physics and Mathematics
Joensuu 2018

Grano Oy
Jyväskylä, 2018
Editors: Tarja Lehto, Pertti Pasanen, Matti Tedre,
Jukka Tuomela, Matti Vornanen

Distribution:
University of Eastern Finland Library / Sales of publications
julkaisumyynti@uef.fi
http://www.uef.fi/kirjasto

ISBN: 978-952-61-2718-7 (print)
ISSNL: 1798-5668
ISSN: 1798-5668
ISBN: 978-952-61-2719-4 (pdf)
ISSNL: 1798-5668
ISSN: 1798-5668

Author's address:	University of Eastern Finland Department of Physics and Mathematics P.O. Box 111 FI-80101 Joensuu FINLAND email: kian.sierra@uef.fi
Supervisors:	Professor Jouni Rättyä. University of Eastern Finland Department of Physics and Mathematics P.O. Box 111 FI-80101 Joensuu FINLAND email: jouni.rattya@uef.fi
	Professor José Ángel Peláez Márquez. Universidad de Málaga Departamento de Análisis Matemático Campus de Teatinos C.P 29071 Málaga SPAIN email: japelaez@uma.es
Reviewers:	Professor Jordi Pau. Universitat de Barcelona Departament de Matemática Aplicada i Analisi C.P. 08007 Barcelona Gran Via 585 SPAIN email: jordi.pau@ub.edu
	Professor Ruhan Zhao. SUNY Brockport Department of Mathematics NY 14420 Brockport UNITED STATES email: rzhao@brockport.edu
Opponent:	Professor Mikael Lindström. Åbo Akademi University Department of Mathematics FI-20500 Åbo Finland email: mikael.lindstrom@abo.fi

This doctoral thesis was jointly supervised in cooperation with the University of Eastern Finland and the University of Malaga (Spain).

Kian Sierra McGettigan
Classical operators on weighted Bergman and mixed norm spaces
Joensuu: University of Eastern Finland, 2018
Publications of the University of Eastern Finland
Dissertations in Forestry and Natural Sciences

ABSTRACT

This thesis introduces new results concerning classical operators of the type $T_\mu : X \to Y$ or $T : X \to L^q_\mu(\mathbb{D})$, where μ is a Borel measure over the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and X, Y are spaces of functions over the unit disk. The space X in most cases will be one of the following; a weighted Bergman spaces A^p_ω, a tent space $T^p_q(h, \omega)$ or a weighted mixed norm space $A^{p,q}_\omega$, where the weight ω is a radial weight that satisfies the doubling condition $\int_r^1 \omega(s)ds \lesssim \int_{\frac{1+r}{2}}^1 \omega(s)ds$, among other possible conditions. Our goal in this thesis will be to characterize properties of the operators such as boundedness, compactness or belonging to a certain Schatten class, in terms of (geometric) conditions over the measure μ.

RESUMEN

Esta tesis contiene resultados originales sobre operadores clásicos de tipo $T_\mu : X \to Y$ o bien $T : X \to L^q_\mu(\mathbb{D})$, donde μ es una medida de Borel definida sobre el disco unidad $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ y X, Y son espacios de funciones, normalmente analíticas, definidas sobre el disco unidad. El espacio X, suele ser alguno de los siguientes espacios de funciones; un espacio de Bergman con pesos A^p_ω, un espacio de tipo tienda $T^p_q(h, \omega)$ o un espacio de norma mixta $A^{p,q}_\omega$, donde el peso ω es radial y satisface la propiedad doblante $\int_r^1 \omega(s)ds \lesssim \int_{\frac{1+r}{2}}^1 \omega(s)ds$. Nuestro objetivo en esta tesis es caracterizar propiedades de estos operadores, tales como la acotación, compacidad o pertenencia a clases de Schatten, en términos de condiciones (geométricas) sobre la medida μ.

MSC 2010: 32A36, 47G10, 42B25, 47B35, 30H20, 46E15, 47B38.
Keywords: Bergman space, Mixed norm space, Tent space, weight, Carleson measure, reproducing kernel, Bergman projection, area operator, Toeplitz operatos, atomic decomposition.

ACKNOWLEDGEMENTS

First of all, I want to thank my supervisors, Professor José Ángel Peláez and Professor Jouni Rättyä, for their advice and guidance during this hard endeavour.

My most sincere thanks to all my high school maths teachers, with a special mention to Don José and Antonio Molina for sharing with me their passion for mathematics. In addition I would like to thank all my maths professors at the University of Málaga, in particular those in the department of analysis for their great work teaching and keeping students engaged. Furthermore I wish to thank the faculty and staff of the Department of Physics and Mathematics of the University of Eastern Finland for taking me in as one of their own and helping me with the transition.

For financial support, I am indebted to the Academy of Finland through the research project of Rättyä #268009 and to the Faculty of Science and Forestry of the University of Eastern Finland, projects #930349, #4900024, #23376. I have also been supported by the Ministerio de Economía y Competitivivad, Spain, projects MTM2014-52865-P and MTM2015-69323-REDT; La Junta de Andalucía, project FQM210.

I wish to thank my family and friends for their support. I thank all my friends at home, for their support and encouragement, alongside all the friends I've made in Joensuu with a special thanks to those from riverside.

Finally I would like to thank my mother for her hard work during these 27 years helping me find my true passion.

Joensuu, February 26, 2018

Kian Sierra McGettigan

LIST OF PUBLICATIONS

This thesis consists of the present review of the author's work in the field of operator theory and the following selection of the author's publications:

I J. Á. Peláez, J. Rättyä and K. Sierra, "Embedding Bergman spaces into tent spaces," *Math. Z.* **281** (2015), no. 3-4, 1215–1237.

II J. Á. Peláez, J. Rättyä and K. Sierra, "Berezin transform and Toeplitz operators on weighted Bergman spaces induced by regular weights" *J. Geom. Anal.* DOI: 10.1007/s12220-017-9837-9 (2017), 1-32.

III J. Á. Peláez, J. Rättyä and K. Sierra, "Atomic decomposition and Carleson measures for weighted mixed norm spaces" Submitted preprint. http://arxiv.org/abs/1709.07239

Throughout the overview, these papers will be referred to by Roman numerals.

AUTHOR'S CONTRIBUTION

The publications selected in this dissertation are original research papers on operator theory.

Paper **I** is a continuation of research done by the other two authors. All authors have made an equal contribution.

Paper **II** is a continuation of research done in Joensuu. All authors have made an equal contribution.

In Paper **III**, all authors have made an equal contribution.

TABLE OF CONTENTS

1 Introduction **1**

2 Notation **3**
2.1 Basic notation ... 3
2.2 Weights ... 4
2.3 Function spaces and some of their properties ... 5
2.3.1 Hardy spaces ... 6
2.3.2 Weighted Bergman spaces ... 6
2.3.3 Weighted mixed norm spaces ... 7
2.3.4 Tent spaces ... 8
2.3.5 Schatten classes ... 8
2.4 Rademacher functions and Khinchine's inequality ... 9

3 Atomic decomposition **11**
3.1 Atomic decomposition in standard Bergman spaces ... 11

4 Operators on spaces of analytic functions **13**
4.1 Carleson measures ... 13
4.1.1 Carleson measures for Hardy spaces ... 13
4.1.2 Carleson measures for standard Bergman spaces ... 13
4.1.3 Carleson measures for weighted Bergman spaces ... 14
4.1.4 Carleson measures for mixed norm spaces ... 17
4.2 Area operators ... 17
4.2.1 Area operators in Hardy spaces ... 17
4.2.2 Area operators in standard Bergman spaces ... 18
4.3 Projections and reproducing kernels ... 20
4.3.1 Projections on standard Bergman spaces ... 20
4.3.2 Toeplitz operator and Berezin transform in standard Bergman spaces ... 22
4.3.3 Projection on weighted Bergman spaces ... 23

5 Summary of papers I-III **25**
5.1 Summary of Paper I ... 25
5.2 Summary of Paper II ... 27
5.3 Summary of Paper III ... 30

BIBLIOGRAPHY **33**

1 Introduction

The main aim of this thesis is to study the boundedness of certain operators of the type $T_\mu : X \to Y$ with a symbol μ or $T : X \to L^q_\mu(\mathbb{D})$. Here μ is a positive Borel measure over the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, X is a space of analytic functions over the unit disk $\mathbb{D}$ and Y is a certain space of functions.

Our main goal is to characterize the boundedness and compactness of the operators T_μ in terms of (geometric) conditions over the measure μ. We will work with a variety of weighted function spaces such as weighted Bergman spaces A^p_ω, tent spaces $T^p_q(h,\omega)$, mixed norm spaces $A^{p,q}_\omega$. Most of our results are given for weights in the class $\widehat{\mathcal{D}}$, these are radial weights $\omega(z) = \omega(|z|)$, that satisfy

$$\int_r^1 \omega(s)ds \lesssim \int_{\frac{1+r}{2}}^1 \omega(s)ds.$$

Basic properties of these weights can be found in [41, 46], additional conditions on the weights might be required in each case. The standard weights $\omega(z) = (1-|z|)^\alpha$ for $\alpha > -1$ all belong to the class $\widehat{\mathcal{D}}$. In particular for every weight $\omega \in \widehat{\mathcal{D}}$, there exists $\alpha(\omega) > -1$ such that $H^p \subset A^p_\omega \subset A^p_\alpha$. Some weights in the class $\widehat{\mathcal{D}}$ satisfy the more restrictive embedding

$$H^p \subset A^p_\omega \subset \cap_{\alpha>-1} A^p_\alpha,$$

which gives us an idea of why in certain problems an approach more common of Hardy spaces is effective.

In order to characterize the boundedness of the operator T_μ we will work in understanding the functions of the given spaces, equivalent norms in these spaces or characterizations of their duals, among other techniques.

In order to study the functions in the given space we will find a suitable family of functions $\{f_n\}$ belonging to the space X, and we will see that all functions of the form $f = \sum c_n f_n$ belong to X, where the sequence $\{c_n\}$ belongs to a given sequence space S, and satisfies the inequality $\|f\|_X \lesssim \|\{c_n\}\|_S$. These inequalities will allow us to discretize the problem at hand. When possible we will also prove that every function in the space X can be written as $f = \sum c_n f_n$.

Another way of identifying the functions in the space X is to find a projection that is bounded onto X. If the projection $P : Z \to X$ is bounded, we can see every $f \in X$ as $f = P(g)$, where $g \in Z$, and Z is a function space of which we already have some information.

Many of the conditions that characterize the boundedness of the given operator T_μ, will consist in proving that μ is a certain Carleson measure. We say μ is a $q-$Carleson measure for the space X if $Id : X \to L^q_\mu(\mathbb{D})$ is bounded. Some of the classical results on Carleson measures were given by Carleson, Duren, Luecking, Jevtic [8, 18, 27, 32] for the spaces H^p, A^p, $A^{p,q}$. Peláez and Rättyä characterized Carleson measure for A^p_ω when $\omega \in \widehat{\mathcal{D}}$ in [42].

The remainder of this survey is organized as follows. In section 2 we give some notation on the subject and recall certain properties on weights and function spaces.

Section 3 contains results on atomic decomposition of standard Bergman spaces A^p_α and mixed norm spaces $A^{p,q}$. In section 4 we discuss some of the previous results on operators, such as Carleson measures, area operators and projections. Finally section 5 summarizes papers **I-III**.

2 Notation

2.1 BASIC NOTATION

We use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$, which depends on certain parameters that will be specified if necessary such that $a \leq Cb$. This constant may change from line to line, and we define $a \gtrsim b$ in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$ we will write $a \asymp b$.

We define the euclidean unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and its boundary $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We denote with $D(a,r) = \{z \in \mathbb{C} : |z - a| < r\}$ the euclidean disk of center $a \in \mathbb{C}$ and radius $r \in (0, \infty)$. The pseudohyperbolic distance is $\varrho(a,z) = \left|\frac{z-a}{1-\overline{a}z}\right|$, and $\Delta(a,r) = \{z \in \mathbb{C} : \varrho(z,a) < r\}$ is the pseudohyperbolic disk with center $a \in \mathbb{D}$ and radius $0 < r < 1$. The hyperbolic distance is defined as $d(a,z) = \frac{1}{2}\log\left(\frac{1+|\varrho(a,z)|}{1-|\varrho(a,z)|}\right)$, and the hyperbolic disk is $\Lambda(a,t) = \{z \in \mathbb{D} : d(a,z) < t\}$ for all $a \in \mathbb{D}$ and $t > 0$.

A sequence $Z = \{z_k\}_{k=0}^{\infty} \subset \mathbb{D}$ is called separated if it is separated in the pseudohyperbolic metric, it is an ε-net for $\varepsilon \in (0,1)$ if $\mathbb{D} = \bigcup_{k=0}^{\infty} \Delta(z_k, \varepsilon)$, and finally it is a δ-lattice if it is a 5δ-net and separated with constant $\delta/5$.

Given two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ we say that a linear operator $T : X \to Y$ is bounded if

$$\|T\|_{(X,Y)} = \sup_{x \in X : \|x\|_X = 1} \{\|T(x)\|_Y\} < \infty.$$

Given $\zeta \in \mathbb{T}$ we define the non-tangential region with vertex at ζ as follows

$$\Gamma(\zeta) = \left\{\xi \in \mathbb{D} : |\theta - \arg(\xi)| < \frac{1}{2}(1 - |\xi|)\right\}, \quad \zeta = e^{i\theta} \in \mathbb{T}.$$

These sets can be generalized to non-tangential regions with vertex at z in the punctured unit disk $\mathbb{D} \setminus \{0\}$,

$$\Gamma(z) = \left\{\xi \in \mathbb{D} : |\theta - \arg(\xi)| < \frac{1}{2}\left(1 - \frac{|\xi|}{r}\right)\right\}, \quad z = re^{i\theta} \in \overline{\mathbb{D}} \setminus \{0\}.$$

The associated tents are defined by $T(\zeta) = \{z \in \mathbb{D} : \zeta \in \Gamma(z)\}$ for all $\zeta \in \mathbb{D} \setminus \{0\}$. When we are working with a radial weight ω we set $\omega(T(0)) = \lim_{r \to 0^+} \omega(T(r))$ to deal with the origin.

The Carleson square $S(I)$ based on an interval $I \subset \mathbb{T}$ is the set $S(I) = \{re^{it} \in \mathbb{D} : e^{it} \in I, 1 - |I| \leq r < 1\}$, where $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{T}$. We associate to each $a \in \mathbb{D} \setminus \{0\}$ the interval $I_a = \{e^{i\theta} : |\arg(ae^{-i\theta})| \leq \frac{1-|a|}{2}\}$, and denote $S(a) = S(I_a)$. For the case $a = 0$ we set $I_0 = \mathbb{T}$, hence $S(0) = \mathbb{D}$.

The polar rectangle associated with an arc $I \subset \mathbb{T}$ is

$$R(I) = \left\{z \in \mathbb{D} : \frac{z}{|z|} \in I,\ 1 - \frac{|I|}{2\pi} \leq |z| < 1 - \frac{|I|}{4\pi}\right\}.$$

Write $z_I = (1 - |I|/2\pi)\xi$, where $\xi \in \mathbb{T}$ is the midpoint of I.

Let Υ denote the family of all dyadic arcs of $\mathbb{T}$. Every arc $I \in \Upsilon$ is of the form

$$I_{n,k} = \left\{ e^{i\theta} : \frac{2\pi k}{2^n} \leq \theta < \frac{2\pi(k+1)}{2^n} \right\},$$

where $k = 0, 1, 2, \ldots, 2^n - 1$ and $n \in \mathbb{N} \cup \{0\}$.

The family $\{R(I) : I \in \Upsilon\}$ consists of pairwise disjoint rectangles whose union covers $\mathbb{D}$. For $I_j \in \Upsilon \setminus \{I_{0,0}\}$, we will write $z_j = z_{I_j}$. For convenience, we associate the arc $I_{0,0}$ with the point $1/2$.

2.2 WEIGHTS

An integrable function $\omega : \mathbb{D} \to [0, \infty)$ is called a weight. We say it is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, and we write $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)ds$. Most of the thesis will focus on radial weights that satisfy the doubling condition $\widehat{\omega}(r) \lesssim \widehat{\omega}(\frac{1+r}{2})$. This class of weights is denoted by $\widehat{\mathcal{D}}$. Given a radial weight ω, we define its associated weight by

$$\omega^\star(z) = \int_{|z|}^1 \omega(s) \log \frac{s}{|z|} s\, ds, \quad z \in \mathbb{D} \setminus \{0\}.$$

The following lemma will show some of the characterizations of the weights in this class.

Lemma 2.2.1. *[41, Lemma 2.1] Let ω be a radial weight. Then the following assertions are equivalent:*

(*i*) $\omega \in \widehat{\mathcal{D}}$;

(*ii*) *There exists $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that*

$$\widehat{\omega}(r) \leq C \left(\frac{1-r}{1-t} \right)^\beta \widehat{\omega}(t), \quad 0 \leq r \leq t < 1;$$

(*iii*) *There exists $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that*

$$\int_0^t \left(\frac{1-t}{1-s} \right)^\gamma \omega(s)\, ds \leq C\widehat{\omega}(t), \quad 0 \leq t < 1;$$

(*iv*) $\omega^\star(z) \asymp \widehat{\omega}(z)(1 - |z|)$, $|z| \to 1^-$;

(*v*) *There exists $\lambda = \lambda(\omega) \geq 0$ such that*

$$\int_{\mathbb{D}} \frac{\omega(z)}{|1 - \overline{\zeta} z|^{\lambda+1}} dA(z) \asymp \frac{\widehat{\omega}(\zeta)}{(1 - |\zeta|)^\lambda}, \quad \zeta \in \mathbb{D};$$

(*vi*) *There exists $C = C(\omega) > 0$ such that $\omega_x = \int_0^1 r^x \omega(r)\, dr$, $x \geq 0$ satisfies $\omega_n \leq C\omega_{2n}$ for $n \in \mathbb{N}$.*

We say a radial weight satisfies the reverse doubling condition if it satisfies one of the following equivalent conditions.

Lemma 2.2.2. *[47] Let ω be a radial weight. Then the following statements are equivalent:*

(i) There exists $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\widehat{\omega}(r) \geq C\widehat{\omega}\left(1 - \frac{1-r}{K}\right) \quad \text{for all } 0 \leq r < 1; \tag{2.1}$$

(ii) There exists $C = C(\omega) > 0$ and $\alpha = \alpha(\omega) > 0$ such that

$$\widehat{\omega}(t) \leq C\left(\frac{1-t}{1-r}\right)^{\alpha} \widehat{\omega}(r), \quad 0 \leq r \leq t < 1;$$

(iii) There exists $C = C(\omega) > 0$ such that

$$\int_r^1 \frac{\widehat{\omega}(s)}{1-s}\,ds \leq C\widehat{\omega}(r).$$

The class of weights that satisfies both the doubling condition and the reverse doubling condition is denoted with $\mathcal{D}$.

We say that a radial weight ω is regular if $\omega \in \widehat{\mathcal{D}}$ and $\frac{\widehat{\omega}(r)}{1-r} \asymp \omega(r)$ for $0 \leq r < 1$, and we denote this class of weights with $\mathcal{R}$. The class of weights $\mathcal{R}$ is contained in $\mathcal{D}$. The standard weights $\omega(r) = (1-r)^{\alpha}$ with $\alpha > -1$ belong to $\mathcal{R}$. We define the class of rapidly increasing weights denoted with $\mathcal{I}$ as those radial weights which are continuous and $\lim_{r\to 1^-} \frac{\widehat{\omega}(r)}{\omega(r)(1-r)} = \infty$. Some examples of weights in the class $\mathcal{I}$ are

$$v_{\alpha}(r) = \frac{1}{(1-r)\left(\log \frac{e}{1-r}\right)^{\alpha}}, \quad 1 < \alpha < \infty.$$

Both these classes $\mathcal{R}$ and $\mathcal{I}$ are subclasses of the class of doubling weights $\widehat{\mathcal{D}}$ by [41].

We denote $\omega(E) = \int_E \omega(\zeta)dA(\zeta)$ where E is a measurable subset of $\mathbb{D}$.

Lemma 2.2.3. *[46, Lemma 1.6]*

(*i*) *if ω is a radial weight, then*

$$\omega^{\star}(z) \asymp \omega(T(z)), \quad |z| \geq \frac{1}{2}.$$

(*ii*) *if $\omega \in \widehat{\mathcal{D}}$, then*

$$\omega(T(z)) \asymp \omega(S(z)), \quad z \in \mathbb{D}.$$

The proof in [46, Lemma 1.6] is restricted to the class $\mathcal{R} \cup \mathcal{I}$ but also works for the class $\widehat{\mathcal{D}}$, as appears in [41, Pag. 55, Eq. 26].

2.3 FUNCTION SPACES AND SOME OF THEIR PROPERTIES

In this section we will define the function spaces that appear throughout this thesis.

2.3.1 Hardy spaces

If $0 < r < 1$ and $f \in \mathcal{H}(\mathbb{D})$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})| dt \right)^{\frac{1}{p}}, \quad 0 < p < \infty, \tag{2.2}$$

$$M_\infty(r,f) = \max_{0 \le t < 2\pi} |f(re^{it})|. \tag{2.3}$$

Since $f \in \mathcal{H}(\mathbb{D})$, the maximum modulus principle and the subharmonicity of $|f|^p$ tells us that the integral means M_p and M_∞ are non-decreasing functions of r.

For $0 < p \le \infty$ we define the Hardy space H^p as the space of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty.$$

Given a function $f : \mathbb{D} \to \mathbb{C}$ we define its non-tangential maximal function as

$$N(f)(e^{i\theta}) = \sup_{z \in \Gamma(e^{i\theta})} |f(z)|, \quad e^{i\theta} \in \mathbb{T}. \tag{2.4}$$

If $f \in H^p$, then $N(f) < \infty$ almost everywhere on $\mathbb{T}$. In particular for $f \in \mathcal{H}(\mathbb{D})$ we have the equivalent norm

$$\|f\|_{H^p} \asymp \|N(f)\|_{L^p(\mathbb{T})}, \tag{2.5}$$

where $L^p(\mathbb{T})$ refers to the standard Lebesgue space on $\mathbb{T}$.

We also recall the Hardy-Stein-Spencer identity, which can be found in [22]

$$\begin{aligned} \|f\|_{H^p}^p &= \frac{1}{2} \int_{\mathbb{D}} \triangle \left(|f|^p\right) \log \frac{1}{|z|} dA(z) + |f(0)|^p \\ &= \frac{p^2}{2} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} dA(z) + |f(0)|^p \\ &\asymp \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \,(1-|z|)\, dA(z) + |f(0)|^p, \end{aligned} \tag{2.6}$$

where $dA(z) = \frac{dxdy}{\pi}$ is the normalized Lebesgue measure on $\mathbb{D}$.

2.3.2 Weighted Bergman spaces

The Bergman spaces were introduced by Bergman [7] and Dzrbashian [16], they worked with the weight $\omega \equiv 1$. Given a weight ω and $0 < p < \infty$, we define the weighted Bergman space A^p_ω as the space of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{A^p_\omega} = \left(\int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) \right)^{\frac{1}{p}} < \infty. \tag{2.7}$$

In particular the Bergman spaces induced by the standard weights $\omega(z) = (1-|z|)^\alpha$ are denoted by A^p_α.

It is clear that for every radial weight ω we have the inclusion $H^p \subset A^p_\omega$. In addition if $\omega \in \widehat{\mathcal{D}}$ we have the inclusion $A^p_\omega \subset A^p_\alpha$ for all $\alpha > \beta(\omega)$, where $\beta(\omega)$ is given by condition (ii) of Lemma 2.2.1.

If ω is a radial weight we can see that the dilated functions $f_r(z) = f(rz)$ approximate the function f in A^p_ω, that is, $\lim_{r\to 1^-} \|f - f_r\|_{A^p_\omega} = 0$. This implies that polynomials are dense in A^p_ω whenever the weight is radial.

For a function in $f \in \mathcal{H}(\mathbb{D})$ we define its non-tangential maximal function as

$$N(f)(\zeta) = \sup_{z\in\Gamma(\zeta)} |f(z)|, \quad \zeta \in \mathbb{D}\setminus\{0\}. \tag{2.8}$$

By applying (2.5) to the dilated functions f_r we obtain the following equivalent norm for A^p_ω whenever ω is a radial weight

$$\|f\|_{A^p_\omega} \asymp \|N(f)\|_{L^p_\omega}. \tag{2.9}$$

The proof can be found in [46, Lemma 4.4], along with additional information on the constants of equivalence. Using (2.6) on the dilated functions f_r together with Fubini's theorem we obtain the following equivalent norm in A^p_ω whenever ω is radial

$$\|f\|^p_{A^p_\omega} = p^2 \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2\omega^\star(z)dA(z) + \omega(\mathbb{D})|f(0)|^p. \tag{2.10}$$

This equivalent norm comes in very handy in the case $p = 2$, since we get an equivalent norm in terms of the derivatives exclusively, the proof of this equivalence can be found in [46, Lemma 4.2]. Another relevant norm in terms of the nth-derivative is the one we can extrapolate from the Littlewood-Paley identity

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} \left(\int_{\Gamma(u)} |f^{(n)}(\zeta)|^2 \left(1 - \left|\frac{\zeta}{u}\right|\right)^{2n-2} \right)^{\frac{p}{2}} \omega(u)dA(u) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p, \tag{2.11}$$

the proof of this equivalence can be found in [46, Lemma 4.2].

2.3.3 Weighted mixed norm spaces

For $0 < p \le \infty$, $0 < q < \infty$ and a radial weight ω, the weighted mixed norm space $A^{p,q}_\omega$ consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^q_{A^{p,q}_\omega} = \int_0^1 M_p^q(r,f)\omega(r)\,dr < \infty.$$

These spaces generalize the weighted Bergman space A^p_ω, since $A^p_\omega = A^{p,q}_\omega$ when $q = p$ and we have the obvious inclusions $A^p_\omega \subset A^{p,q}_\omega$ when $q < p$. The mixed norm spaces were introduced by Benedek and Panzone in [6].

For $0 < p \le \infty$, $0 < q < \infty$ and a radial weight ω, the space $L^{p,q}_\omega$ consists of measurable functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^q_{L^{p,q}_\omega} = \int_0^1 M_p^q(r,f)\omega(r)\,dr < \infty.$$

2.3.4 Tent spaces

The tent spaces were introduced by Coifman, Meyer and Stein in [12] and later work appeared involving these spaces, such as the work done by Cohn and Verbitsky in [11]. In their context, they work on the upper half-space $\mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1} : t > 0\}$. We will first introduce the tent spaces on the unit disk, as described by Cohn in [10] and we will follow with the generalization of these spaces that appears in [42]. We will divide the traditional tent spaces in to three groups as follows.

For $0 < p < \infty$, the tent spaces T^p_∞ consist of those (equivalence classes of) measurable functions $u : \mathbb{D} \to \mathbb{C}$ such that

$$\|u\|_{T^p_\infty} = \left(\int_{\mathbb{T}} \left(N(u)(e^{i\theta}) \right)^p d\theta \right)^{\frac{1}{p}} < \infty.$$

Given $0 < p, q < \infty$, the tent spaces T^p_q consist of those (equivalence classes of) measurable functions $u : \mathbb{D} \to \mathbb{C}$ such that

$$\|u\|_{T^p_q} = \left(\int_{\mathbb{T}} \left(\int_{\Gamma(e^{i\theta})} |u(z)|^q \frac{dA(z)}{1-|z|} \right)^{\frac{p}{q}} d\theta \right)^{\frac{1}{q}} < \infty.$$

For $0 < q < \infty$, the tent spaces T^∞_q consist of those (equivalence classes of) measurable functions $u : \mathbb{D} \to \mathbb{C}$ such that

$$\|u\|_{T^\infty_q} = \sup_{I \subset \mathbb{T}} \left(\frac{1}{|I|} \int_{T(I)} |u(z)|^q \frac{dA(z)}{1-|z|} \right)^{\frac{1}{q}} < \infty.$$

For $0 < q < \infty$, a positive Borel measure ν on $\mathbb{D}$ finite on compact sets, and a function $f : \mathbb{D} \to \mathbb{C}$, we denote $A^q_{q,\nu}(f)(\zeta) = \int_{\Gamma(\zeta)} |f(z)|^q \, d\nu(z)$ and $A_{\infty,\nu}(f)(\zeta) = \nu\text{-ess}\sup_{z \in \Gamma(\zeta)} |f(z)|$, for all $\zeta \in \mathbb{D}$. For $0 < p < \infty$, $0 < q \leq \infty$ and $\omega \in \widehat{\mathcal{D}}$, the tent space $T^p_q(\nu,\omega)$ consists of the (ν-equivalence classes of) ν-measurable functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{T^p_q(\nu,\omega)} = \|A_{q,\nu}(f)\|_{L^p_\omega} < \infty.$$

For $0 < q < \infty$ we define

$$C^q_{q,\nu}(f)(\zeta) = \sup_{a \in \Gamma(\zeta)} \frac{1}{\omega(T(a))} \int_{T(a)} |f(z)|^q \omega(T(z)) \, d\nu(z), \quad \zeta \in \mathbb{D} \setminus \{0\}.$$

A quasi-norm in the tent space $T^\infty_q(\nu,\omega)$ is defined by $\|f\|_{T^\infty_q(\nu,\omega)} = \|C_{q,\nu}(f)\|_{L^\infty}$.

2.3.5 Schatten classes

Let H be a separable Hilbert space. For any non-negative integer n, the n:th singular value of a bounded operator $T : H \to H$ is defined by

$$\lambda_n(T) = \inf \{ \|T - R\| : \text{rank}(R) \leq n \},$$

where $\|\cdot\|$ denotes the operator norm. It is clear that

$$\|T\| = \lambda_0(T) \geq \lambda_1(T) \geq \lambda_2(T) \geq \cdots \geq 0.$$

For $0 < p < \infty$, the Schatten p-class $\mathcal{S}_p(H)$ consists of those compact operators $T : H \to H$ whose sequence of singular values $\{\lambda_n\}_{n=0}^{\infty}$ belongs to the space ℓ^p of p-summable sequences. For $1 \le p < \infty$, the Schatten p-class $\mathcal{S}_p(H)$ is a Banach space with respect to the norm $|T|_p = \|\{\lambda_n\}_{n=0}^{\infty}\|_{\ell^p}$. Therefore all finite rank operators belong to every $\mathcal{S}_p(H)$, and the membership of an operator in $\mathcal{S}_p(H)$ measures in some sense the size of the operator. We refer to [17] and [54, Chapter 1] for more information about $\mathcal{S}_p(H)$.

2.4 RADEMACHER FUNCTIONS AND KHINCHINE'S INEQUALITY

The Rademacher functions are defined as $\varphi_n(t) = \operatorname{sgn}(\sin(2^n \pi t))$, $0 \le t \le 1$, these functions form an orthonormal system over the interval $[0, 1]$. In particular the first function is

$$\varphi_1(t) = \begin{cases} 1, & 0 < t < \frac{1}{2}; \\ -1, & \frac{1}{2} < t < 1; \\ 0, & t = 0, \frac{1}{2}, 1. \end{cases}$$

Given a sequence $\{a_n\} \in \ell^2$, we define the function $\Phi(t) = \sum_n a_n \varphi_n(t), \quad 0 \le t \le 1$, which is well defined almost everywhere. These functions satisfy the following estimate for all $0 < p < \infty$.

$$\left(\int_0^1 |\Phi(t)|^p \, dt \right)^{\frac{1}{p}} \asymp \left(\sum_n |a_n|^2 \right)^{\frac{1}{2}}. \tag{2.12}$$

This estimate is known as Khinchine's inequality and was proven in [29]. For more information on the topic of Rademacher functions and Khinchine's inequality the reader may check [19, Appendix 1] or [55, Chapter 5.8].

3 Atomic decomposition

3.1 ATOMIC DECOMPOSITION IN STANDARD BERGMAN SPACES

The idea behind an atomic decomposition of a Banach space X is to obtain a representation of every function in the space of the form $f = \sum_n c_n g_n$, where the functions $g_n \in X$ are a fixed sequence which are called atoms. These atoms satisfy certain properties and $\{c_n\}$ is a sequence which belongs to a certain sequence space ℓ. We will also have the norm estimate $\|f\|_X \asymp \|\{c_n\}\|_\ell$. Obtaining an atomic decomposition for a certain Banach space X helps the study of certain properties of operators defined on X, such as boundedness and compactness, among others. In the case of a standard Bergman space the following theorem is known.

Theorem 3.1.1. *[13] Suppose $p > 0$, $\alpha > -1$ and*

$$b > \max\left(1, \frac{1}{p}\right) + \frac{\alpha+1}{p}.$$

Then there exists a constant $\sigma > 0$ such that for any r-lattice $\{a_k\}$ in the Bergman metric, where $0 < r < \sigma$, the function space A^p_α consists exactly of functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^{(pb-2-\alpha)/p}}{(1-z\overline{a_k})^b}, \tag{3.1}$$

where $\{c_k\} \in \ell^p$, the series in (3.1) converges in norm A^p_α, and $\|f\|_{A^p_\alpha}$ is comparable to

$$\inf\left\{\|c\|_{\ell^p} : c = \{c_k\} \text{ satisfies (3.1)}\right\}.$$

The proof of this result was given by Coifman and Rochberg in [13] by extending the proof of certain results given by Amar [5].

As an extension of Theorem 3.1.1, Ricci and Taibleson [49] obtained an atomic decomposition for the mixed norm spaces $A^{p,q}$, but first in order to state the result we shall introduce the space of doubled indexed sequences. For $0 < p, q < \infty$ we define the set $\ell^{p,q}$ of doubled indexed sequences $\lambda_{j,k}$ such that $\|\lambda\|_{\ell^{p,q}} = \left(\sum_j \left(\sum_k |\lambda_{j,k}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty$, and in a similar fashion we define the sets $\ell^{\infty,q}$, $\ell^{p,\infty}$ and $\ell^{\infty,\infty}$ by using $\sup|\cdot|$. We have the following inclusions between these spaces

$$\begin{aligned} \ell^{p,q} &\subset \ell^{r,q}, \quad 0 < p \le r \le \infty, \\ \ell^{p,q} &\subset \ell^{p,s}, \quad 0 < q \le s \le \infty. \end{aligned} \tag{3.2}$$

We will work with these spaces of sequences with one of the indexes finite. The following theorem from Nakamura [36, Theorem 1] describes the duals of these spaces.

Theorem 3.1.2. *[36, Theorem 1] For any* $0 < p \leq \infty$*, we define* $p' = \frac{p}{p-1}$ *if* $1 < p < \infty$*,* $p' = 1$ *if* $p = \infty$ *and* $p' = \infty$ *if* $0 < p \leq 1$*. Then*

$$\sup\left\{\left|\sum_{i,j} c_{i,j} b_{i,j}\right| : \|c\|_{p,q} = 1\right\} = \|b\|_{p',q'} .$$

The next theorem was originally proven in [49, Theorem 1.5] in the context of $\mathbb{R}^2_+$. Given a sequence $\{z_k\}$ we re-index it in the following way for each $j \in \mathbb{N} \cup \{0\}$ we set $\{z_{i,j}\}$ as those $\{z_k\}$ such that $2^{-j} \leq 1 - |z_{i,j}| < 2^{-j+1}$, $i = 0, \dots, 2^j - 1$.

Theorem 3.1.3. *Define the operator* S *on double indexed sequences by*

$$S(\{a_{i,j}\})(z) = \sum_n a_{i,j} \frac{(1 - |z_{i,j}|^2)^{M - \frac{1}{p} - \frac{1}{q}}}{(1 - \overline{z_{i,j}} z)^M},$$

where $M > \max\left\{1, \frac{1}{q}\right\} + \frac{1}{p}$*. Then* S *is bounded from* $\ell^{p,q}$ *to* $A^{p,q}$ *whenever* $\{z_{i,j}\}$ *is separated.*

Some results on atomic decomposition in weighted Bergman spaces A^p_ω were given by Constantin [14] when the weight ω belongs to the class of Békollé-Bonami weights.

4 Operators on spaces of analytic functions

4.1 CARLESON MEASURES

Given a space X of analytic functions in $\mathbb{D}$ and a positive Borel measure μ on $\mathbb{D}$, we say μ is a q-Carleson measure for X if the identity operator $Id : X \to L^q_\mu(\mathbb{D})$ is bounded.

Before we dive into Carleson measures for weighted Bergman spaces with $\omega \in \widehat{\mathcal{D}}$, we will state the results for Hardy and standard Bergman spaces. Recall that there exists $\alpha = \alpha(\omega, p)$ such that $H^p \subset A^p_\omega \subset A^p_\alpha$. These inclusions tell us that the conditions that characterize Carleson measures for Hardy spaces will be necessary conditions for μ to be a Carleson measure for A^p_ω. In an analogous manner the conditions that characterize Carleson measures for standard Bergman spaces A^p_α will be sufficient conditions (for α big enough) for μ to be a Carleson measure for A^p_ω.

4.1.1 Carleson measures for Hardy spaces

The characterization of q-Carleson measures in Hardy spaces is divided into two main cases. The case $q = p$ was proven by Carleson in [8], and later on was extended by Duren [18] to the case $p \leq q$ by using test functions, a covering Lemma and the maximal operator given by $M(\phi)(z) = \sup_{I: I_z \subset I} \frac{1}{|I|} \int_I |\phi(\zeta)| d\zeta$.

Theorem 4.1.1. *[18] Let $0 < p \leq q < \infty$, and μ a positive Borel measure on $\mathbb{D}$. Then μ is a q-Carleson measure for H^p if and only if*

$$\sup_{I \subset \mathbb{T}} \frac{\mu\left(S(I)\right)}{|I|^{\frac{q}{p}}} < \infty.$$

For the case where $0 < q < p < \infty$ Luecking [31] uses Khinchine's inequality (2.12), subharmonicity, and the theory of tent spaces T^p_q related to Hardy spaces to obtain the following result.

Theorem 4.1.2. *[31] Let $0 < q < p < \infty$, and μ a positive Borel measure on $\mathbb{D}$. Then μ is a q-Carleson measure for H^p if and only if the function*

$$A_\mu(e^{i\theta}) = \int_{\Gamma(e^{i\theta})} \frac{d\mu(\zeta)}{1 - |\zeta|^2}, \quad e^{i\theta} \in \mathbb{T},$$

belongs to $L^{\frac{p}{p-q}}(\mathbb{T})$.

4.1.2 Carleson measures for standard Bergman spaces

The first results that characterize Carleson measures on standard Bergman spaces were given by Hastings [24] offering a description for $1 \leq p \leq q < \infty$, and Oleinik and Pavlov [37] giving a characterization for $0 < p \leq q < \infty$.

Theorem 4.1.3. *[37] Let $0 < p \leq q < \infty$, and μ a positive Borel measure on $\mathbb{D}$. Then the following assertions are equivalent:*

(*i*) *μ is a q-Carleson measure for A^p_α;*

(*ii*)

$$\sup_{a\in\mathbb{D}} \frac{\mu\left(S(a)\right)}{\left(1-|a|^2\right)^{(2+\alpha)\frac{q}{p}}} < \infty;$$

(*iii*) *For every $0 < r < 1$*

$$\sup_{a\in\mathbb{D}} \frac{\mu\left(\Delta(a,r)\right)}{\left(1-|a|^2\right)^{(2+\alpha)\frac{q}{p}}} < \infty.$$

The condition $\sup_{a\in\mathbb{D}} \frac{\mu(\Delta(a,r))}{(1-|a|^2)^{(2+\alpha)\frac{q}{p}}} < \infty$ is a necessary condition for μ to be a q-Carleson measure for A^p_α for any $0 < p,q < \infty$. To prove this it suffices to estimate the $L^q_\mu(\mathbb{D})$ norm of the normalized reproducing kernels. However it is not a sufficient condition when $0 < q < p < \infty$. In this case, a characterization of Carleson measures for standard Bergman spaces was proven by Luecking [32], using Rademacher functions and Khinchine's inequality (2.12), together with the atomic decomposition of standard Bergman spaces given by Theorem 3.1.1 and the properties of $\delta-$lattices.

Theorem 4.1.4. *[32] Let $0 < q < p < \infty$, and μ a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(*i*) *μ is a q-Carleson measure for A^p_α;*

(*ii*) *For every $0 < r < 1$ the function $\widehat{\mu_r}(z) = \frac{\mu(\Delta(z,r))}{(1-|z|)^{2+\alpha}}$ belongs to $L_\alpha^{\frac{p}{p-q}}(\mathbb{D})$;*

(*iii*) *The function $\widetilde{\mu}(z) = \frac{\mu(S(z))}{(1-|z|)^{2+\alpha}}$ belongs to $L_\alpha^{\frac{p}{p-q}}(\mathbb{D})$.*

Since these theorems work for the standard weights $\omega(z) = (1-|z|^2)^\alpha$, and $\omega(S(z)) \asymp \omega(\Delta(z,r)) \asymp (1-|z|^2)^{2+\alpha}$, it is reasonable to try to find a characterization of Carleson measure for a more general ω, by replacing $(1-|z|)^{2+\alpha}$ with either $\omega(\Delta(z,r))$ or $\omega(S(z))$ in the previous theorems.

4.1.3 Carleson measures for weighted Bergman spaces

In this section we discuss Carleson measures on weighted Bergman spaces A^p_ω with $\omega \in \widehat{\mathcal{D}}$. In order to study the Carleson measures in A^p_ω we need to define the following Hörmander-type weighted maximal function. Given a positive Borel measure μ on $\mathbb{D}$ and $\alpha > 0$, we define the weighted maximal function

$$M_{\omega,\alpha}(\mu)(z) = \sup_{z\in S(a)} \frac{\mu(S(a))}{\left(\omega\left(S(a)\right)\right)^\alpha}, \quad z \in \mathbb{D}. \tag{4.1}$$

In the case $\alpha = 1$, we omit it from the notation and write $M_{\omega,1}(\mu) = M_\omega(\mu)$. If ϕ is a non-negative function we write $M_{\omega,\alpha}(\phi)$ as $M_{\omega,\alpha}$ acting over the measure $\phi\omega dA$.

Theorem 4.1.5. *[41] Let $0 < p \leq q < \infty$ and $0 < \gamma < \infty$ such that $p\gamma > 1$. Let $\omega \in \widehat{\mathcal{D}}$ and μ be a positive Borel measure on $\mathbb{D}$. Then $[M_\omega((\cdot)^{\frac{1}{\gamma}})]^\gamma : L^p_\omega \to L^q_\mu(\mathbb{D})$ is bounded if and only if $M_{\omega,q/p}(\mu) \in L^\infty$. Moreover,*

$$\|[M_\omega((\cdot)^{\frac{1}{\gamma}})]^\gamma\|^q_{\left(L^p_\omega, L^q(\mu)\right)} \asymp \|M_{\omega,q/p}(\mu)\|_{L^\infty}.$$

The boundedness of this operator plays a big role in characterizing the Carleson measures for the weighted Bergman space A^p_ω.

Theorem 4.1.6. *[42] Let $0 < p, q < \infty$, $\omega \in \widehat{\mathcal{D}}$ and μ be a positive Borel measure on $\mathbb{D}$.*

(a) If $p > q$, the following conditions are equivalent:

(i) μ is a q-Carleson measure for A^p_ω;

(ii) The function

$$B_\mu(z) = \int_{\Gamma(z)} \frac{d\mu(\zeta)}{\omega(T(\zeta))}, \quad z \in \mathbb{D} \setminus \{0\},$$

belongs to $L_\omega^{\frac{p}{p-q}}(\mathbb{D})$;

(iii) $M_\omega(\mu) \in L_\omega^{\frac{p}{p-q}}(\mathbb{D})$.

(b) μ is a p-Carleson measure for A^p_ω if and only if $M_\omega(\mu) \in L^\infty(\mathbb{D})$.

(c) If $q > p$, the following conditions are equivalent:

(i) μ is a q-Carleson measure for A^p_ω;

(ii) $M_{\omega,q/p}(\mu) \in L^\infty(\mathbb{D})$;

(iii) $z \mapsto \dfrac{\mu\left(\Delta(z,r)\right)}{\left(\omega\left(S(z)\right)\right)^{\frac{q}{p}}}$ belongs to $L^\infty(\mathbb{D})$ for any fixed $r \in (0,1)$.

There are many results about Carleson measures on A^p_ω for other classes of weights. We will state some of them such as the characterization of Carleson measures on weighted Bergman spaces induced by rapidly decreasing weights $\mathcal{W}$ defined in [38], which include the family of weights

$$\omega_\alpha(r) = \exp\left(\frac{-1}{(1-r)^\alpha}\right), \quad \alpha > 0. \tag{4.2}$$

Theorem 4.1.7. *[38, Theorem 1] Let $\omega \in \mathcal{W}$ and let μ be a finite positive Borel measure on $\mathbb{D}$.*

(i) Let $0 < p \leq q < \infty$. Then μ is a q-Carleson measure for A^p_ω if and only if for each sufficiently small $\delta > 0$ we have

$$\sup_{a \in \mathbb{D}} \frac{1}{\tau(a)^{\frac{2q}{p}}} \int_{D(a,\delta\tau(a))} \omega(z)^{\frac{-q}{p}} \, d\mu(z) < \infty.$$

(ii) Let $0 < q < p < \infty$. Then μ is a q-Carleson measure for A^p_ω if and only if for each sufficiently small $\delta > 0$ the function

$$z \to \frac{1}{\tau(z)^{\frac{2q}{p}}} \int_{D(z,\delta\tau(z))} \omega(\zeta)^{\frac{-q}{p}} d\mu(\zeta) \in L^{\frac{p}{p-q}}(\mathbb{D}).$$

Here $\tau : \mathbb{D} \to (0,1)$ is a radial function which depends on ω and has the property that it decreases to 0 as $|z| \to 1^-$. For more information on the class $\mathcal{W}$ and the function τ we refer to [38].

There is literature on the characterization of Carleson measures for A^p_ω when ω is a non-radial weight, like the results given by O.Constantin [15] for the Békolle weight class. A weight ω is said to belong to the Békolle class $B_{p_0}(\eta)$, where $p_0 > 1$ and $\eta > -1$ if there exists $k > 0$ such that

$$\int_{S(a)} \omega(z) dA_\eta(z) \left(\int_{S(a)} \omega(z)^{-\frac{p_0'}{p_0}} dA_\eta(z) \right)^{\frac{p_0}{p_0'}} \leq k \left(\int_{S(a)} dA_\eta(z) \right)^{p_0}, \quad a \in \mathbb{D}, \tag{4.3}$$

where $dA_\eta(z) = (\eta+1)(1-|z|)^\eta dA(z)$.

Theorem 4.1.8. *[15, Theorem 3.1] Let ω be a weight such that $\frac{\omega(z)}{(1-|z|)^\eta}$ belongs to $B_{p_0}(\eta)$ for some $p_0 > 0$, $\eta > -1$. Consider a positive finite Borel measure μ on $\mathbb{D}$ and assume $q \geq p > 0$, $n \in \mathbb{N}$. Then there exists a constant $C > 0$ such that*

$$\left(\int_{\mathbb{D}} |f^{(n)}(z)|^q d\mu(z) \right)^{\frac{1}{q}} \leq C \|f\|_{A^p_\omega} \tag{4.4}$$

if and only if μ satisfies

$$\mu(D(a, \alpha(1-|a|))) \leq C'(1-|a|^2)^{nq} \left(\int_{D(a,\alpha(1-|a|))} \omega(z) dA(z) \right)^{\frac{q}{p}} \tag{4.5}$$

for some constant $C' > 0$ independent of a, and for some $\alpha \in (0,1)$.

Theorem 4.1.9. *[15, Theorem 3.2] Let ω be a weight such that $\frac{\omega(z)}{(1-|z|)^\eta}$ belongs to $B_{p_0}(\eta)$ for some $p_0 > 0$, $\eta > -1$. Consider a positive finite Borel measure μ on $\mathbb{D}$ and assume $p > q > 0$, $n \in \mathbb{N}$. Then there exists a constant $C > 0$ such that*

$$\left(\int_{\mathbb{D}} |f^{(n)}(z)|^q d\mu(z) \right)^{\frac{1}{q}} \leq C \|f\|_{A^p_\omega} \tag{4.6}$$

if and only if the function

$$a \to \frac{\mu(D(a, \alpha(1-|a|)))}{(1-|a|^2)^{nq} \int_{D(a,\alpha(1-|a|))} \omega(z) dA(z)} \tag{4.7}$$

belongs to $L_\omega^{\frac{p}{p-q}}$, for some $\alpha \in (0,1)$.

4.1.4 Carleson measures for mixed norm spaces

Jevtić gives a characterization of Carleson measures for weighted mixed norm spaces in the half-plane $\mathbb{R}^2_+$. We reindex the the family $\{R(I) : I \in \Upsilon\}$ as $\{R_{i,j}\}$, $j \in \mathbb{N} \cup \{0\}$, $i = 0, \ldots, 2^j - 1$.

Theorem 4.1.10. *[27, Theorem 3.3] Let μ be a positive Borel measure on $\mathbb{D}$, $0 < s < p < \infty$ and $0 < t < q < \infty$. Then there exists $C > 0$ such that*

$$\left(\sum_j \left(\sum_i \int_{R_{i,j}} |f(z)|^s d\mu(z) \right)^{\frac{t}{s}} \right)^{\frac{1}{t}} \leq C \|f\|_{A^{p,q}}, \tag{4.8}$$

if and only if

$$\sum_j \left(\sum_i \mu(R_{i,j})^{\frac{1}{s}} 2^{ju\left(\frac{1}{p}+\frac{1}{q}\right)} \right)^{\frac{v}{u}} < \infty, \tag{4.9}$$

where $\frac{1}{u} = \frac{1}{s} - \frac{1}{p}$ and $\frac{1}{v} = \frac{1}{t} - \frac{1}{q}$.

Note that in the case where $s = t$ and $p = q$, Jevtić gives a characterization of the s-Carleson measures for A^p. Luecking extended the result given by Jevtić by eliminating the restriction over the coefficients and by adding a continuous characterization.

Theorem 4.1.11. *[32, Theorem 2] Let $0 < p, q, s < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) There exists $C > 0$ such that $\|f^{(n)}\|_{L^s_\mu} \leq C\|f\|_{A^{p,q}}$ for all $f \in A^{p,q}$;

(ii) The sequence $\left\{ \mu(R_{i,j}) 2^{sj(n+\frac{1}{p}+\frac{1}{q})} \right\}$ belongs to $\ell^{\left(\frac{p}{s}\right)', \left(\frac{q}{s}\right)'}$;

(iii) For all $0 < r < 1$ the function k_r belongs to $L^{\left(\frac{p}{s}\right)', \left(\frac{q}{s}\right)'}$, where

$$k_r(z) = \begin{cases} \frac{\mu(\Delta(z,r))}{(1-|z|)^{2+sn}}, & s < \min\{p,q\}; \\ \frac{\mu(\Delta(z,r))}{(1-|z|)^{1+\frac{s}{p}+sn}}, & p \leq s < q; \\ \frac{\mu(\Delta(z,r))}{(1-|z|)^{1+\frac{s}{q}+sn}}, & q \leq s < p; \\ \frac{\mu(\Delta(z,r))}{(1-|z|)^{\frac{s}{p}+\frac{s}{q}+sn}}, & \max\{p,q\} \leq s. \end{cases}$$

In the proof of this theorem, Luecking uses Theorem 3.1.3 given by Ricci and Taibleson [49], Rademacher functions and Khinchine's inequality (2.12), and an equivalent norm for the functions in $A^{p,q}$.

4.2 AREA OPERATORS

4.2.1 Area operators in Hardy spaces

As Ahern and Bruna showed in [1], a function $f \in H^p$ if and only if for a given $\alpha > 0$

$$\int_{\mathbb{T}} \left(\int_{\Gamma(e^{i\theta})} (1-r)^{2\alpha} \, |D^{\alpha}(z)|^2 \, \frac{dA(z)}{(1-|z|)^2} \right)^{\frac{p}{2}} d\theta < \infty, \tag{4.10}$$

where $D^{\alpha} f$ is the radial fractional derivative of order α. In the paper [10] Cohn extracted the following area operator

$$\mathcal{A}_{\mu}(f)(e^{i\theta}) = \int_{\Gamma(e^{i\theta})} |f(z)| \frac{d\mu(z)}{1-|z|}, \tag{4.11}$$

and characterized its boundedness from H^p to $L^p(\mathbb{T})$ in terms of the measure μ.

Theorem 4.2.1. *[10, Theorem 1] Let $0 < p < \infty$ and μ a positive Borel measure on $\mathbb{D}$. Then $\mathcal{A}_{\mu}$ is bounded from H^p to $L^p(\mathbb{T})$ if and only if μ is a p-Carleson measure for H^p.*

In a later work Gong, Lou, and Wu characterized in [23] the boundedness of $\mathcal{A}_{\mu}$ from H^p to $L^q(\mathbb{T})$ with the following results.

Theorem 4.2.2. *[23, Theorem 3.1] Let $0 < p \leq q < \infty$ and μ a non-negative measure on $\mathbb{D}$. Then $\mathcal{A}_{\mu}$ is bounded from H^p to $L^q(\mathbb{T})$ if and only if μ is a $\left(1 + \frac{1}{p} - \frac{1}{q}\right)$-Carleson measure for H^1.*

In order to prove this result, they use the equivalent norm given in (2.5), the test functions $f_a(z) = \frac{(1-|a|)^m}{(1-\bar{a}z)^{m+\frac{1}{p}}}$, the boundedness of the Riesz projection from $L^q(\mathbb{T})$ to H^q when $q > 1$ and Calderon-Zygmund decompositions among other techniques.

Theorem 4.2.3. *[23, Theorem 3.2] Let $1 \leq q < p \leq \infty$ and μ a non-negative measure on $\mathbb{D}$. Then $\mathcal{A}_{\mu}$ is bounded from H^p to $L^q(\mathbb{T})$ if and only if $\widehat{\mu}(\zeta) = \int_{\Gamma(\zeta)} \frac{d\mu(z)}{1-|z|}$ for $\zeta \in \mathbb{T}$ belongs to $L^{\frac{pq}{p-q}}(\mathbb{T})$.*

For this result they relied again on the equivalent norm (2.5), the characterization of Carleson measures in Hardy spaces described in Theorem 4.1.2 and some estimates involving the non-tangential maximal operator N.

4.2.2 Area operators in standard Bergman spaces

Given a non-negative Borel measure μ on $\mathbb{D}$ and $\tau > 0$, the area operator $\mathcal{A}_{\mu}^{\tau}$ on the Bergman space A_{α}^{p} is defined as

$$\mathcal{A}_{\mu}^{\tau}(\zeta) = \int_{\Gamma_{\tau}(\zeta)} |f(z)| \frac{d\mu(z)}{1-|z|}, \quad \zeta \in \mathbb{T},$$

where the non-tangential regions $\Gamma_{\tau}(\zeta)$ as defined by Wu in [51] are

$$\Gamma_{\tau}(\zeta) = \{z \in \mathbb{D} : |z-\zeta| < (1+\tau)(1-|z|)\}, \quad \zeta \in \mathbb{T}.$$

He characterized the boundedness of the area operator $\mathcal{A}_{\mu}^{\tau}$ in terms of the growth of the α-density function $\rho_{\alpha}(\mu)(z,t) = \frac{\mu(\Lambda(z,t))}{|\Lambda(z,t)|_{\alpha}}$, where $|\Lambda(z,t)|_{\alpha} = \int_{\Lambda(z,t)|}(1-|w|^2)^{\alpha} dA(w)$.

Theorem 4.2.4. *[51, Theorem 1] Suppose $\alpha > -1$, $0 < p \le q < \infty$ and $0 < p \le 1$. For a non-negative Borel measure μ on $\mathbb{D}$ the following statements are equivalent:*

(i) $\mathcal{A}_\mu^1$ *is bounded from* A_α^p *to* $L^q(\mathbb{T})$*;*

(ii) There exists $\tau > 0$ *such that* $\mathcal{A}_\mu^\tau$ *is bounded from* A_α^p *to* $L^q(\mathbb{T})$*;*

(iii) There exists $\delta > 0$ *such that the* α*-density function* $\rho_\alpha(\mu)$ *satisfies*

$$\rho_\alpha(\mu)(z,\delta) \lesssim (1-|z|)^{(\alpha+2)(\frac{1}{p}-1)+1-\frac{1}{q}}, \quad z \in \mathbb{D};$$

(iv) For any fixed $t > 0$ *the* α*-density function* $\rho_\alpha(\mu)$ *satisfies*

$$\rho_\alpha(\mu)(z,t) \lesssim (1-|z|)^{(\alpha+2)(\frac{1}{p}-1)+1-\frac{1}{q}}, \quad z \in \mathbb{D}.$$

In this proof, Wu groups the conditions (i) with (ii) and (iii) with (iv). In one direction he used the test functions $f_a(z) = \frac{(1-|a|^m)}{(1-\bar{a}z)^{m+\frac{\alpha+2}{p}}}$, which are uniformly bounded in the A_α^p norm and essentially constant in $\Delta(a,r)$, and the boundedness of $\mathcal{A}_\mu^\tau$ from A_α^p to $L^q(\mathbb{T})$, to obtain the growth estimate on $\rho_\alpha(\mu)$. For the reverse implication he uses a suitable δ-lattice on $\mathbb{D}$, the subharmonicity of the functions in A_α^p, together with some geometric arguments and the conditions on p and q.

Theorem 4.2.5. *[51, Theorem 2] Suppose $\alpha > -1$ and $1 < p \le q < \infty$. For a non-negative Borel measure μ on $\mathbb{D}$ the following statements are equivalent:*

(i) $\mathcal{A}_\mu^1$ *is bounded from* A_α^p *to* $L^q(\mathbb{T})$*;*

(ii) There exists $\tau > 0$ *such that* $\mathcal{A}_\mu^\tau$ *is bounded from* A_α^p *to* $L^q(\mathbb{T})$*;*

(iii) There exists $\delta > 0$ *such that the* α*-density function* $\rho_\alpha(\mu)(z,\delta)$ *satisfies that the measure*

$$\rho_\alpha(\mu)(z,\delta)^{\frac{p}{p-1}} dA_\alpha(z)$$

is a $\frac{p(q-1)}{q(p-1)}$*-Carleson measure for* H^1*;*

(iv) For any fixed $t > 0$ *the* α*-density function* $\rho_\alpha(\mu)(z,t)$ *satisfies that the measure*

$$\rho_\alpha(\mu)(z,t)^{\frac{p}{p-1}} dA_\alpha(z)$$

is a $\frac{p(q-1)}{q(p-1)}$*-Carleson measure for* H^1*.*

Here, in order to prove this theorem, Wu uses the duality of $L^q(\mathbb{T})$ and ℓ^q when $q > 1$, a characterization of Carleson measures for Hardy spaces, together with Rademacher functions and Khinchine's inequality (2.12), δ-lattices and their properties and the test functions given by an atomic decomposition like the one given in (3.1), among other tools.

Theorem 4.2.6. *[51, Theorem 3] Suppose $\alpha > -1$ and $1 \le q < p < \infty$. For a non-negative Borel measure μ on $\mathbb{D}$ the following are equivalent:*

(i) $\mathcal{A}^1_\mu$ *is bounded from* A^p_α *to* $L^q(\mathbb{T})$*;*

(ii) *There exists* $\tau > 0$ *such that* $\mathcal{A}^\tau_\mu$ *is bounded from* A^p_α *to* $L^q(\mathbb{T})$*;*

(iii) *There exists* $\delta, \tau > 0$ *such that the* α*-density function* $\rho_\alpha(\mu)(z,\delta)$ *satisfies the property*

$$\int_{\Gamma_\tau(\zeta)} (\rho_\alpha(\mu)(z,\delta))^{\frac{p}{p-1}} \frac{dA_\alpha(z)}{1-|z|} \in L^{\frac{q(p-1)}{p-q}}(\mathbb{T});$$

(iv) *For any fixed* $t, \tau > 0$ *the* α*-density function* $\rho_\alpha(\mu)(z,t)$ *satisfies the property*

$$\int_{\Gamma_\tau(\zeta)} (\rho_\alpha(\mu)(z,t))^{\frac{p}{p-1}} \frac{dA_\alpha(z)}{1-|z|} \in L^{\frac{q(p-1)}{p-q}}(\mathbb{T}).$$

To prove this Theorem Wu uses the duality of L^q, Khinchine's inequality (2.12), the properties of δ-lattices and the Poisson integral.

Theorem 4.2.7. *[51, Theorem 4] Suppose* $\alpha > -1$, $q < p < \infty$, $0 < q < 1$, μ *a non-negative Borel measure on* $\mathbb{D}$*. Then* $\mathcal{A}_\mu$ *is bounded from* A^p_α *to* $L^q(\mathbb{T})$ *if and only if for any fixed* $\tau > 0$, δ*-lattice* $\{z_j\}$ *in* $\mathbb{D}$*, any* $\{a_j\} \in \ell^p$*, the* α*-density function* $\rho_\alpha(\mu)$ *satisfies*

$$\int_{\mathbb{T}} \left(\sum_{j: z_j \in \Gamma_\tau(\zeta)} |a_j| \frac{\rho_\alpha(\mu)(z_j,\delta)}{(1-|z_j|)^{(\alpha+2)(\frac{1}{p}-1)+1}} \right)^q |d\zeta| \lesssim \|\{a_j\}\|^q_{\ell^p}.$$

In order to prove the Theorem Wu uses the test functions $F_t(z) = \sum_j a_j r_j(t) \frac{(1-|z_j|^m)}{(1-\overline{z_j}z)^{m+\frac{\alpha+2}{p}}}$, which satisfy the estimate $\|F_t\|_{A^p_\alpha} \lesssim \|\{a_j\}\|_{\ell^p}$, together with a generalization of Khinchine's inequality given by Kalton [28] and the properties of δ-lattices.

4.3 PROJECTIONS AND REPRODUCING KERNELS

4.3.1 Projections on standard Bergman spaces

For each $a \in \mathbb{D}$, we define the point evaluation functionals

$$\begin{aligned} L_a : A^p_\alpha &\to \mathbb{C}, \\ f &\mapsto f(a), \end{aligned}$$

which is bounded for each $a \in \mathbb{D}$. Focusing on the case $p = 2$, A^2_α is a Hilbert space with the scalar product

$$\langle f, g \rangle_{A^2_\alpha} = \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z).$$

Since L_a is a bounded and linear functional, due to Riesz representation theorem [50, Theorem 4.12, pag.80], there exists $K^\alpha_a \in A^2_\alpha$ such that

$$f(a) = L_a(f) = \langle f, K^\alpha_a \rangle_{A^2_\alpha} = \int_{\mathbb{D}} f(z)\overline{K^\alpha_a(z)} dA_\alpha(z), \quad f \in A^2_\alpha, \quad a \in \mathbb{D}. \tag{4.12}$$

The family of analytic functions $\{K^\alpha_a\}_{a \in \mathbb{D}}$ is called the reproducing kernel of A^2_α. The reproducing kernel K^α_a has an explicit formula, which can be obtained as follows.

Theorem 4.3.1. *[54, Theorem 4.19] Let $\{e_n\}_{n=0}^{\infty}$ an ortonormal basis of A^2_α, then*

$$K_z^\alpha(\xi) = \sum_{n=0}^{\infty} e_n(\xi)\overline{e_n(z)}, \quad z, \xi \in \mathbb{D}. \tag{4.13}$$

Next, we take a a concrete basis of A^2_α. For each $n \in \mathbb{N} \cup \{0\}$, the canonical basis is given by

$$e_n(z) = \frac{z^n}{\left(2(\alpha+1)\int_0^1 s^{2n+1}(1-s^2)^\alpha\, ds\right)^{\frac{1}{2}}}, \quad z \in \mathbb{D}.$$

So, by Theorem 4.3.1,

$$K_z^\alpha(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n \overline{z}^n}{2(\alpha+1)\int_0^1 r^{2n+1}(1-r^2)^\alpha dr}, \quad z, \xi \in \mathbb{D}. \tag{4.14}$$

Then by using some properties of the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt, \quad x > 0,$$

we can rewrite the reproducing kernel K_z^α in the following way.

Corollary 4.3.2. *[54, Corollary 4.20] For $\alpha > -1$ the reproducing kernel of A^2_α is given by*

$$K_z^\alpha(\xi) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+2)}{\Gamma(\alpha+2)\Gamma(n+1)}(\xi\overline{z})^n = \frac{1}{(1-\overline{z}\xi)^{\alpha+2}}, \quad z, \xi \in \mathbb{D}.$$

This together with (4.12) allows us to write the following reproducing formula.

Theorem 4.3.3. *[54, Proposition 4.23] Let $\alpha > -1$, then*

$$f(a) = \int_{\mathbb{D}} \frac{f(z)}{(1-\overline{z}a)^{2+\alpha}} dA_\alpha(z), \quad f \in A^1_\alpha, a \in \mathbb{D}.$$

Since A^2_α is a closed subspace of $L^2_\alpha(\mathbb{D})$, there exists an orthogonal projection from $L^2_\alpha(\mathbb{D})$ to A^2_α, which we shall denote by P_α. Since P_α is an orthogonal projection it satisfies $P^2 = P$, and it is self-adjoint

$$\langle P(f), g\rangle_{L^2_\alpha(\mathbb{D})} = \langle f, P(g)\rangle_{L^2_\alpha(\mathbb{D})}.$$

If $\phi \in L^2_\alpha(\mathbb{D})$ then $P_\alpha(\phi) \in A^2_\alpha$, and the following formula holds

$$\begin{aligned} P_\alpha(\phi)(z) &= \langle P_\alpha(\phi), K_z^\alpha\rangle_{L^2_\alpha(\mathbb{D})} = \langle \phi, P_\alpha(K_z^\alpha)\rangle_{L^2_\alpha(\mathbb{D})} \\ &= \langle \phi, K_z^\alpha\rangle_{L^2_\alpha(\mathbb{D})} = \int_{\mathbb{D}} \phi(\xi)\overline{K_z^\alpha(\xi)} dA_\alpha(\xi) \\ &= \int_{\mathbb{D}} \frac{\phi(\xi)}{(1-\overline{\xi}z)^{\alpha+2}} dA_\alpha(\xi). \end{aligned}$$

If $\phi \in L^p_\alpha(\mathbb{D})$ for $p \geq 2$ the previous formula is also true since $L^p_\alpha(\mathbb{D}) \subset L^2_\alpha(\mathbb{D})$. The projection P_α also makes sense for $\phi \in L^1_\alpha(\mathbb{D})$ since for $z \in \mathbb{D}$ we have

$$\left|\int_{\mathbb{D}} \frac{\phi(\xi)}{(1-\overline{\xi}z)^{\alpha+2}} dA_\alpha(\xi)\right| \leq \frac{1}{(1-|z|)^{2+\alpha}} \int_{\mathbb{D}} |\phi(\xi)|\, dA_\alpha(\xi) = \frac{\|\phi\|_{L^1_\alpha}}{(1-|z|)^{2+\alpha}}, \quad z \in \mathbb{D}.$$

Next we are going to characterize the boundedness of the projection P_γ on the corresponding $L^p_\alpha(\mathbb{D})$ spaces, alongside with analogous result for the maximal projection $P^+_\gamma(\phi)(z) = \int_{\mathbb{D}} \frac{|\phi(\xi)|}{|1-\overline{\xi}z|^{2+\gamma}} dA_\gamma(\xi)$, which is a sublinear operator. Forelli and Rudin proved in [21] the case $\alpha = 0$.

Theorem 4.3.4. *[54, Theorem 4.24] Let $\gamma, \alpha > -1$ and $1 \leq p < \infty$. Then the following statements are equivalent:*

(a) $P_\gamma : L^p_\alpha(\mathbb{D}) \to A^p_\alpha$ *is bounded;*

(b) $P^+_\gamma : L^p_\alpha(\mathbb{D}) \to L^p_\alpha(\mathbb{D})$ *is bounded;*

(c) $p(\gamma+1) > \alpha + 1$.

Now that we have studied the boundedness of the Bergman projection, we will use this to identify $\left(A^p_\alpha\right)^*$, the dual of A^p_α, with $A^{p'}_\alpha$, for $\alpha > -1$ and $1 < p < \infty$.

Theorem 4.3.5. *[54, Theorem 4.25] Let $1 < p < \infty$, and $\alpha > -1$. Then the following statements are true:*

(i) Every $g \in A^{p'}_\alpha$ defines a functional $T_g \in \left(A^p_\alpha\right)^*$ as follows

$$T_g(f) = \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z),$$

with $\|T_g\| \leq \|g\|_{A^{p'}_\alpha}$.

(ii) For every $T \in \left(A^p_\alpha\right)^*$, there exists $g \in A^{p'}_\alpha$ such that

$$T(f) = T_g(f) = \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z), \tag{4.15}$$

with $\|g\|_{A^{p'}_\alpha} \leq C\|T\|$, where $C = C(p)$ is a constant.

For more information on the boundedness of Bergman projection P_α the reader may check [20, 44, 54].

4.3.2 Toeplitz operator and Berezin transform in standard Bergman spaces

Some of the early results with respect to Toeplitz operators were introduced by McDonald and Sundberb [34] and Coburn [9]. Given $\beta > -1$ and a positive Borel measure μ on $\mathbb{D}$, define the Toeplitz operator T^β_μ as follows:

$$T^\beta_\mu f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{2+\beta}}\, d\mu(w) = \int_{\mathbb{D}} f(w) K^\beta_z(w)\, d\mu(w), \quad z \in \mathbb{D}.$$

Pau and Zhao [39] characterized the boundedness of the Toeplitz operator in $\mathbb{C}^n$ from $A^{p_1}_{\alpha_1}$ to $A^{p_2}_{\alpha_2}$. The following result is mostly due to Luecking ([30] and [32]) for the case $\alpha = 0$.

Theorem 4.3.6. *[39, Theorem B] Let $\alpha > -1$, $0 < q < p < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) *μ is a $q-$Carleson measure for A^p_α;*

(ii) *The function*

$$\widehat{\mu}_r(z) = \frac{\mu(\Delta(z,r))}{(1-|z|^2)^{(2+\alpha)}}, \quad z \in \mathbb{D},$$

is in $L^{p/(p-q)}_\alpha(\mathbb{D})$ for any (some) fixed $r \in (0,1)$;

(iii) *For any r-lattice $\{a_k\}$ and $D_k = \Delta(a_k, r)$, the sequence*

$$\{\mu_k\} = \left\{ \frac{\mu(D_k)}{(1-|a_k|^2)^{(2+\alpha)\frac{q}{p}}} \right\}$$

belongs to $\ell^{p/(p-q)}$ for any (some) fixed $r \in (0,1)$;

(iv) *For any $s > 0$, the Berezin-type transform $B_{s,\alpha}(\mu)$ belongs to $L^{\frac{p}{p-q}}_\alpha(\mathbb{D})$.*

Furthermore, with $\lambda = q/p$, one has

$$\|\widehat{\mu}_r\|_{L^{\frac{p}{p-q}}_\alpha(\mathbb{D})} \asymp \|\{\mu_k\}\|_{\ell^{p/(p-q)}} \asymp \|B_{s,\alpha}(\mu)\|_{L^{\frac{p}{p-q}}_\alpha(\mathbb{D})} \asymp \|Id\|^q_{(A^p_\alpha, L^q_\mu(\mathbb{D}))}.$$

Here, for a positive measure μ, the Berezin-type transform $B_{s,\alpha}(\mu)$ is

$$B_{s,\alpha}(\mu)(z) = \int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\overline{w}z|^{2+s+\alpha}}\, d\mu(w), \quad z \in \mathbb{D}.$$

For more information on the Berezin transform we refer to the book [25]. For additional information with respect to the Toeplitz operator, see [54].

4.3.3 Projection on weighted Bergman spaces

From now on, we assume that the norm convergence in the Bergman space A^2_ω implies the uniform convergence on compact subsets, then the point evaluations L_z (at the point $z \in \mathbb{D}$) are bounded linear functionals on A^2_ω. Therefore, there are reproducing kernels $B^\omega_z \in A^2_\omega$ with $\|L_z\| = \|B^\omega_z\|_{A^2_\omega}$ such that

$$L_z f = f(z) = \langle f, B^\omega_z \rangle_{A^2_\omega} = \int_{\mathbb{D}} f(\zeta)\, \overline{B^\omega_z(\zeta)}\, \omega(\zeta)\, dA(\zeta), \quad f \in A^2_\omega. \tag{4.16}$$

In a similar fashion as in (4.13) for any orthonormal basis of A^2_ω

$$B^\omega_z(\zeta) = \sum_{n=0}^{\infty} e_n(\zeta)\overline{e_n(z)}, \quad z, \zeta \in \mathbb{D}.$$

When ω is radial, we can use the standard orthonormal basis $\{z^j/\sqrt{2\omega_{2j+1}}\}$, $j \in \mathbb{N} \cup \{0\}$, of A^2_ω to obtain the following formula for the Bergman reproducing kernels

$$B^\omega_z(\zeta) = \sum_{n=0}^{\infty} \frac{(\zeta\overline{z})^n}{2\omega_{2n+1}}, \quad z, \zeta \in \mathbb{D}. \tag{4.17}$$

Even if ω is a radial weight, the reproducing kernels B_z^ω, $z \in \mathbb{D}$, do not necessarily have the good properties that the standard reproducing kernels B_z^α have. The fact that it is not always possible to obtain a closed formula for the reproducing kernel B_z^ω makes it harder to know relevant information about the kernel such as; its behaviour in pseudohyperbolic bounded regions, the existence of zeros or a norm estimate. One of the main issues with these reproducing kernels B_z^ω is the existence of zeros, which can appear even in radial weights with apparently good properties as Zeytuncu [52] and Perälä [48] proved.

Theorem 4.3.7. *[52, Theorem 1.5] There exists a radial weight ω on $\mathbb{D}$, comparable to 1, such that the reproducing kernel B_a^ω has zeros.*

Since A^2_ω is a closed subspace of $L^2_\omega(\mathbb{D})$, we may consider the orthogonal Bergman projection P_ω from L^2_ω to A^2_ω. This projection is usually called the Bergman projection and it is given by the following formula

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B_z^\omega(\zeta)}\,\omega(\zeta)dA(\zeta), \quad z \in \mathbb{D}. \tag{4.18}$$

The maximal Bergman projection is the following sublinear operator

$$P_\omega^+(f)(z) = \int_{\mathbb{D}} |f(\zeta)B_z^\omega(\zeta)|\,\omega(\zeta)dA(\zeta), \quad z \in \mathbb{D}.$$

Theorem 4.3.8. *[45, Theorem 12] Let $1 < p < \infty$ and $\omega \in \mathcal{R}$. Then the following statements are true:*

(i) $P_\omega^+ : L^p_\omega(\mathbb{D}) \to L^p_\omega(\mathbb{D})$ *is bounded. In particular,* $P_\omega : L^p_\omega(\mathbb{D}) \to A^p_\omega$ *is bounded.*

(ii) $P_\omega : L^\infty(\mathbb{D}) \to \mathcal{B}$ *is bounded.*

The first part of this theorem is a direct consequence of [44, Theorem 3]. The $L^p_\omega(\mathbb{D})$ norm estimates of the Bergman reproducing kernels were obtained using estimates on the moments ω_n by Peláez and Rättyä.

Theorem 4.3.9. *[44, Theorem 1] Let $0 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ and $N \in \mathbb{N} \cup \{0\}$. Then the following assertions hold:*

(i) $M_p^p\left(r, (B_a^\omega)^{(N)}\right) \asymp \int_0^{|a|r} \frac{dt}{\widehat{\omega}(t)^p(1-t)^{p(N+1)}}, \quad r, |a| \to 1^-.$

(ii) If $v \in \widehat{\mathcal{D}}$, then

$$\| (B_a^\omega)^{(N)} \|^p_{A^p_v} \asymp \int_0^{|a|} \frac{\widehat{v}(t)}{\widehat{\omega}(t)^p(1-t)^{p(N+1)}}\,dt, \quad |a| \to 1^-.$$

When P_ω is bounded on L^p_ω we can use it to obtain the dual of A^p_ω, hence under the same conditions as in Theorem 4.3.8 Peláez and Rättyä obtained the following result.

Corollary 4.3.10. *[44, Corollary 2] Let $1 < p < \infty$ and $\omega \in \mathcal{R}$. Then the following equivalences hold under the A^2_ω pairing: $(A^p_\omega)^\star \simeq A^{p'}_\omega$ and $(A^1_\omega)^\star \simeq \mathcal{B}$.*

5 Summary of papers I-III

5.1 SUMMARY OF PAPER I

In this paper we characterize the embedding $A^p_\omega \subset T^q_s(\nu,\omega)$ in terms of Carleson measures for A^p_ω. This result can be interpreted as an additional characterization of Carleson measures for the space A^p_ω. We add the technical condition $\nu(\{0\}) = 0$ because the tents $\Gamma(z)$ are not defined for $z = 0$, this condition does not carry any real restriction.

Theorem 5.1.1. *Let $0 < p,q,s < \infty$ such that $1 + \frac{s}{p} - \frac{s}{q} > 0$, $\omega \in \widehat{\mathcal{D}}$ and let ν be positive Borel measure on $\mathbb{D}$, finite on compact sets, such that $\nu(\{0\}) = 0$. Write $\nu_\omega(\zeta) = \omega(T(\zeta))\,d\nu(\zeta)$ for all $\zeta \in \mathbb{D}\setminus\{0\}$. Then the following assertions hold:*

(i) $I_d : A^p_\omega \to T^q_s(\nu,\omega)$ is bounded if and only if ν_ω is a $(p+s-\frac{ps}{q})$-Carleson measure for A^p_ω. Moreover,

$$\|I_d\|^s_{A^p_\omega \to T^q_s(\nu,\omega)} \asymp \|I_d\|^{p+s-\frac{ps}{q}}_{A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\nu_\omega}}.$$

(ii) $I_d : A^p_\omega \to T^q_s(\nu,\omega)$ is compact if and only if $I_d : A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\nu_\omega}(\mathbb{D})$ is compact.

We can generalize this theorem by extracting the following area operator and studying its boundedness. For $0 < s < \infty$, the generalized area operator induced by positive measures μ and v on $\mathbb{D}$ is defined by

$$G^v_{\mu,s}(f)(z) = \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu(\zeta)}{v(T(\zeta))}\right)^{\frac{1}{s}}, \quad z \in \mathbb{D}\setminus\{0\}.$$

Minkowski's inequality shows that $G^v_{\mu,s}$ is sublinear if $s \geq 1$. This is not the case for $0 < s < 1$, but instead we have $\left(G^v_{\mu,s}(f+g)\right)^s \leq \left(G^v_{\mu,s}(f)\right)^s + \left(G^v_{\mu,s}(g)\right)^s$. Write μ^ω_v for the positive measure such that

$$d\mu^\omega_v(z) = \frac{\omega(T(z))}{v(T(z))}\,d\mu(z)$$

for μ-almost every $z \in \mathbb{D}$. Fubini's theorem shows that

$$\begin{aligned}
\|G^v_{\mu,s}(f)\|^s_{L^s_\omega(\mathbb{D})} &= \int_{\mathbb{D}} \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu(\zeta)}{v(T(\zeta))}\right) \omega(z)\,dA(z) \\
&= \int_{\mathbb{D}} \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))}\right) \omega(z)\,dA(z) \\
&= \int_{\mathbb{D}\setminus\{0\}} |f(\zeta)|^s \left(\frac{1}{\omega(T(\zeta))}\int_{T(\zeta)} \omega(z)\,dA(z)\right) d\mu^\omega_v(\zeta) \\
&= \int_{\mathbb{D}\setminus\{0\}} |f(\zeta)|^s\,d\mu^\omega_v(\zeta) = \|f\|^s_{L^s_{\mu^\omega_v}(\mathbb{D})} - |f(0)|^s \mu^\omega_v(\{0\}),
\end{aligned} \tag{5.1}$$

and hence $G^v_{\mu,s} : A^p_\omega \to L^s_\omega(\mathbb{D})$ is bounded if and only if μ^ω_v is an s-Carleson measure for A^p_ω. In order to study this operator the equivalent norm given by (2.9) will play an important role.

Theorem 5.1.2. *Let $0 < p, q, s < \infty$ such that $s > q - p$, $\omega \in \widehat{\mathcal{D}}$ and let μ, v be positive Borel measures on $\mathbb{D}$ such that $\mu(\{z \in \mathbb{D} : v(T(z)) = 0\}) = 0 = \mu(\{0\})$. Then the following assertions hold:*

(i) μ^ω_v is a q-Carleson measure for A^p_ω if and only if $G^v_{\mu,s} : A^p_\omega \to L_\omega^{\frac{ps}{p+s-q}}(\mathbb{D})$ is bounded. Moreover,

$$\|G^v_{\mu,s}\|^s_{A^p_\omega \to L_\omega^{\frac{ps}{p+s-q}}(\mathbb{D})} \asymp \|I_d\|^q_{A^p_\omega \to L^q_{\mu^\omega_v}(\mathbb{D})}.$$

(ii) $I_d : A^p_\omega \to L^q_{\mu^\omega_v}(\mathbb{D})$ is compact if and only if $G^v_{\mu,s} : A^p_\omega \to L_\omega^{\frac{ps}{p+s-q}}(\mathbb{D})$ is compact.

In order to prove the previous theorems we need to estimate the norm of the identity operator $Id : A^p_\omega \to L^q_\mu(\mathbb{D})$. The equivalence between the conditions of the following theorem was proven by Peláez and Rättyä in [42, Theorem 1].

Theorem 5.1.3. *Let $0 < p, q < \infty$, $\omega \in \widehat{\mathcal{D}}$ and let μ be a positive Borel measure on $\mathbb{D}$. Further, let $dh(z) = dA(z)/(1-|z|^2)^2$ denote the hyperbolic measure.*

(i) If $p \le q$, then μ is a q-Carleson measure for A^p_ω if and only if $\sup_{a\in\mathbb{D}} \frac{\mu(S(a))}{\omega(S(a))^{\frac{q}{p}}} < \infty$. Moreover,

$$\|I_d\|^q_{A^p_\omega \to L^q_\mu(\mathbb{D})} \asymp \sup_{a\in\mathbb{D}} \frac{\mu(S(a))}{\omega(S(a))^{\frac{q}{p}}}.$$

(ii) If $p \le q$, then $I_d : A^p_\omega \to L^q_\mu(\mathbb{D})$ is compact if and only if

$$\lim_{|a|\to 1^-} \frac{\mu(S(a))}{(\omega(S(a)))^{q/p}} = 0. \tag{5.2}$$

(iii) If $q < p$, then the following conditions are equivalent:

(a) $I_d : A^p_\omega \to L^q_\mu(\mathbb{D})$ is compact;

(b) $I_d : A^p_\omega \to L^q_\mu(\mathbb{D})$ is bounded;

(c) The function

$$B^\omega_\mu(z) = \int_{\Gamma(z)} \frac{d\mu(\zeta)}{\omega(T(\zeta))}, \quad z \in \mathbb{D}\setminus\{0\},$$

belongs to $L_\omega^{\frac{p}{p-q}}(\mathbb{D})$;

(d) For each fixed $r \in (0,1)$, the function

$$\Phi^\omega_\mu(z) = \Phi^\omega_{\mu,r}(z) = \int_{\Gamma(z)} \frac{\mu(\Delta(\zeta,r))}{\omega(T(\zeta))}\, dh(\zeta), \quad z \in \mathbb{D}\setminus\{0\},$$

belongs to $L_\omega^{\frac{p}{p-q}}(\mathbb{D})$;

(e) *For each sufficiently large $\lambda = \lambda(\omega) > 1$, the function*

$$\Psi_\mu^\omega(z) = \Psi_{\mu,\lambda}^\omega(z) = \int_{\mathbb{D}} \left(\frac{1-|\zeta|}{|1-\bar{z}\zeta|} \right)^\lambda \frac{d\mu(\zeta)}{\omega(T(\zeta))}, \quad z \in \mathbb{D},$$

belongs to $L_\omega^{\frac{p}{p-q}}(\mathbb{D})$.

Moreover,

$$\begin{aligned}\|I_d\|^q_{A^p_\omega \to L^q_\mu(\mathbb{D})} &\asymp \|M_\omega(\mu)\|_{L_\omega^{\frac{p}{p-q}}(\mathbb{D})} \asymp \|B_\mu^\omega\|_{L_\omega^{\frac{p}{p-q}}(\mathbb{D})} + \mu(\{0\}) \\ &\asymp \|\Psi_\mu^\omega\|_{L_\omega^{\frac{p}{p-q}}(\mathbb{D})} \asymp \|\Phi_\mu^\omega\|_{L_\omega^{\frac{p}{p-q}}(\mathbb{D})} + \mu(\{0\}).\end{aligned} \tag{5.3}$$

To prove the compactness of these operators, we use the following lemma, the proof of which can be obtained by adapting the proof in [19, Lemma 1 p. 21]

Lemma 5.1.4. *Let ν be a positive Borel measure on $\mathbb{D}$ and $0 < p < \infty$. If $\{\varphi_n\}_{n=0}^\infty \subset L^p_\nu(\mathbb{D})$ and $\varphi \in L^p_\nu(\mathbb{D})$ satisfy $\lim_{n\to\infty} \|\varphi_n\|_{L^p_\nu(\mathbb{D})} = \|\varphi\|_{L^p_\nu(\mathbb{D})}$ and $\lim_{n\to\infty} \varphi_n(z) = \varphi(z)$ ν-a.e. on $\mathbb{D}$, then $\lim_{n\to\infty} \|\varphi_n - \varphi\|_{L^p_\nu(\mathbb{D})} = 0$.*

We can use Theorem 5.1.2 to study the boundedness of other operators such as the following integral operators, also called Volterra type operators. Given $g \in \mathcal{H}(\mathbb{D})$ we define the integral operator

$$T_g(f)(z) = \int_0^z g'(\zeta) f(\zeta)\, d\zeta, \quad z \in \mathbb{D},$$

acting on $\mathcal{H}(\mathbb{D})$. Some results of this operator in the context of Hardy spaces are due to Aleman and Cima [2] and Aleman and Siskakis [3], while in the context of Bergman spaces we find the results given by Aleman and Siskakis in [4]. This type of integral operators have been extensively studied during recent decades and have interesting connections with other areas of mathematical analysis, see [41, 46] and the references therein.

Theorem 5.1.5. *Let $0 < p, q < \infty$ such that $q > \frac{2p}{2+p}$ and $\omega \in \widehat{\mathcal{D}}$. Let $g \in \mathcal{H}(\mathbb{D})$ and denote $d\mu_g(z) = |g'(z)|^2 \omega^\star(z)\, dA(z)$. Then $T_g : A^p_\omega \to A^q_\omega$ is bounded (resp. compact) if and only if $I_d : A^p_\omega \to L_{\mu_g}^{p+2-\frac{2p}{q}}(\mathbb{D})$ is bounded (resp. compact).*

This result is a direct consequence of Theorem 5.1.2 with $\nu = \omega dA$, $\mu = \mu_g$, and $s = 2$ together with the equivalent norm given in (2.10).

5.2 SUMMARY OF PAPER II

In this paper we work with weights ω which belong to either $\mathcal{R}$ or $\widehat{\mathcal{D}}$. The Bergman projection P_ω is given by

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta) \overline{B_z^\omega(\zeta)} \omega(\zeta)\, dA(\zeta), \quad z \in \mathbb{D},$$

where B_z^ω are the reproducing kernels stated in (4.17). Recently, those regular weights ω and ν for which $P_\omega : L^p_\nu(\mathbb{D}) \to L^p_\nu(\mathbb{D})$ is bounded were characterized in terms of Bekollé-Bonami type conditions [44]. In this paper we consider operators which are natural extensions of the projection P_ω. For a positive Borel measure μ on $\mathbb{D}$, the Toeplitz operator associated with μ is

$$\mathcal{T}_\mu(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B_z^\omega(\zeta)}\, d\mu(\zeta).$$

If $d\mu = \Phi\omega dA$ for a non-negative function Φ, then write $\mathcal{T}_\mu = \mathcal{T}_\Phi$ so that $\mathcal{T}_\Phi(f) = P_\omega(f\Phi)$. The operator $\mathcal{T}_\Phi$ has been extensively studied since the seventies [9,34,53]. Luecking was probably the one who introduced Toeplitz operators $\mathcal{T}_\mu$ with measures as symbols in [30], where he provides, among other things, a description of Schatten class Toeplitz operators $\mathcal{T}_\mu : A^2_\alpha \to A^2_\alpha$ in terms of an ℓ^p-condition involving a hyperbolic lattice of $\mathbb{D}$.

Before presenting our main results we will need some additional results on the reproducing kernels B_z^ω, apart from those given in Theorem 4.3.9. First we will estimate the norm in the Bloch space and H^∞.

Lemma 5.2.1. *Let $\omega \in \widehat{\mathcal{D}}$. Then*

$$\|B_z^\omega\|_{\mathcal{B}} \asymp \frac{1}{\omega(S(z))} \asymp \|B_z^\omega\|_{H^\infty}, \quad z \in \mathbb{D}.$$

Here the Bloch norm is given by $\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|)|f'(z)|$. Additionally to prove the main result of this section we will need the following pointwise estimates of the reproducing Bergman kernels B_a^ω in certain sets induced by a point $a \in \mathbb{D}$.

Lemma 5.2.2. *Let $\omega \in \widehat{\mathcal{D}}$. Then there exists $r = r(\omega) \in (0,1)$ such that $|B_a^\omega(z)| \asymp B_a^\omega(a)$ for all $a \in \mathbb{D}$ and $z \in \Delta(a,r)$.*

In this first result we used the definition of B_a^ω in terms of a basis and the norm estimates $\|B_a^\omega\|_{A^2_\omega}$ from Theorem 4.3.9.

Lemma 5.2.3. *Let $\omega \in \widehat{\mathcal{D}}$. Then there exists constants $c = c(\omega) > 0$ and $\delta = \delta(\omega) \in (0,1]$ such that*

$$|B_a^\omega(z)| \geq \frac{c}{\omega(S(a))}, \quad z \in S(a_\delta), \quad a \in \mathbb{D}\setminus\{0\}. \tag{5.4}$$

To prove this result we use estimates of the derivative of B_a^ω, the fact that $B_a^\omega(a_\delta) = B^\omega_{\sqrt{|aa_\delta|}}(\sqrt{|aa_\delta|})$ and Lemma 2.2.1.

These pointwise estimates will allow us to avoid some of the issues of weighted reproducing Bergman kernels B_a^ω such as the possible existence of zeros in $\mathbb{D}$.

Theorem 5.2.4. *Let $1 < p \leq q < \infty$, $\omega \in \mathcal{R}$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ *is bounded;*

(ii) $\dfrac{\widetilde{\mathcal{T}_\mu}(\cdot)}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}-1}} \in L^\infty(\mathbb{D})$;

(iii) μ *is a* $\frac{s(p+q')}{pq'}$*-Carleson measure for* A^s_ω *for some (equivalently for all)* $0 < s < \infty$;

(iv) $\frac{\mu(S(\cdot))}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}}} \in L^\infty(\mathbb{D})$.

Moreover,

$$\|\mathcal{T}_\mu\|_{A^p_\omega \to A^q_\omega} \asymp \left\| \frac{\widetilde{\mathcal{T}}_\mu(\cdot)}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}-1}} \right\|_{L^\infty(\mathbb{D})} \asymp \|Id\|^{\frac{s(p+q')}{pq'}}_{A^s_\omega \to L_\mu^{\frac{s(p+q')}{pq'}}(\mathbb{D})} \asymp \left\| \frac{\mu(S(\cdot))}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}}} \right\|_{L^\infty(\mathbb{D})}.$$

In order to prove this result we use the norm estimates given in Theorem 4.3.9, the pointwise estimate Lemma 5.2.3 and the duality of the space A^q_ω given in Corollary 4.3.10.

Theorem 5.2.5. *Let $1 < p \le q < \infty$, $\omega \in \mathcal{R}$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ *is compact;*

(ii) $\lim_{|z|\to 1^-} \frac{\widetilde{\mathcal{T}}_\mu(z)}{\omega(S(z))^{\frac{1}{p}+\frac{1}{q'}-1}} = 0$;

(iii) $Id : A^s_\omega \to L_\mu^{\frac{s(p+q')}{pq'}}(\mathbb{D})$ *is compact for some (equivalently for all)* $0 < s < \infty$;

(iv) $\lim_{|I|\to 0} \frac{\mu(S(I))}{\omega(S(I))^{\frac{1}{p}+\frac{1}{q'}}} = 0.$

To prove this result we proceed in a similar fashion as in Theorem 5.2.4, together with common techniques usually employed to study the compactness of concrete operators, and the weak convergence of $\frac{B^\omega_z}{\|B^\omega_z\|_{A^p_\omega}} \to 0$ in A^p_ω as $|z| \to 1^-$.

Proposition 5.2.6. *Let $1 < p < \infty$, $\omega \in \mathcal{R}$ and $\{z_j\}_{j=1}^\infty \subset \mathbb{D} \setminus \{0\}$ be a separated sequence. Then*

$$F = \sum_{j=1}^\infty c_j \frac{B^\omega_{z_j}}{\|B^\omega_{z_j}\|_{A^p_\omega}} \in A^p_\omega \tag{5.5}$$

with $\|F\|_{A^p_\omega} \lesssim \|\{c_j\}_{j=1}^\infty\|_{\ell^p}$ for all $\{c_j\}_{j=1}^\infty \in \ell^p$.

For us to prove the reverse case when $1 < q < p < \infty$, we will need the test functions F given by the result above, which is obtained using Theorem 4.3.10, the reproducing formula for $B^\omega_{z_j}$ and the subharmonicity of these functions.

Theorem 5.2.7. *Let $1 < q < p < \infty$, $0 < r < 1$, $\omega \in \mathcal{R}$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ *is compact;*

(ii) $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ *is bounded;*

(iii) $\widehat{\mu}_r(\cdot) = \frac{\mu(\Delta(\cdot,r))}{\omega(\Delta(\cdot,r))} \in L_\omega^{\frac{pq}{p-q}}(\mathbb{D})$;

(iv) μ *is a* $\left(p+1-\frac{p}{q}\right)$*-Carleson measure for* A^p_ω;

(v) $Id : A^p_\omega \to L^{p+1-\frac{p}{q}}_\mu(\mathbb{D})$ *is compact;*

(vi) $\widetilde{\mathcal{T}_\mu} \in L^{\frac{pq}{p-q}}_\omega(\mathbb{D})$.

Moreover,

$$\|\mathcal{T}_\mu\|_{A^p_\omega \to A^q_\omega} \asymp \|\widehat{\mu}_r\|_{L^{\frac{qp}{p-q}}_\omega(\mathbb{D})} \asymp \|Id\|^{p+1-\frac{p}{q}}_{A^p_\omega \to L^{p+1-\frac{p}{q}}_\mu} \asymp \|\widetilde{\mathcal{T}_\mu}\|_{L^{\frac{qp}{p-q}}_\omega(\mathbb{D})}.$$

The tools used to prove this result, are the characterization of Carleson measures for A^p_ω given by Theorem 4.1.6, the test functions given by Proposition 5.2.6, Rademacher functions and Khinchine's inequality and the boundedness of the sublinear operator P^+_ω given by Theorem 4.3.8.

Theorem 5.2.8. *Let* $0 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ *and* μ *be a positive Borel measure on* $\mathbb{D}$. *Then the following statements are equivalent:*

(i) $\mathcal{T}_\mu \in \mathcal{S}_p(A^2_\omega)$;

(ii) $\sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{\omega^\star(z_j)}\right)^p < \infty$;

(iii) $\frac{\mu(\Delta(\cdot,r))}{\omega^\star(\cdot)}$ *belongs to* $L^p\left(\frac{dA}{(1-|\cdot|)^2}\right)$ *for some* $0 < r < 1$.

Moreover, if $\omega \in \mathcal{R}$ *such that* $\frac{(\omega^\star(z))^p}{(1-|z|)^2}$ *is also a regular weight, then* $\mathcal{T}_\mu \in \mathcal{S}_p(A^2_\omega)$ *if and only if* $\widetilde{\mathcal{T}_\mu} \in L^p_{\omega/\omega^\star}(\mathbb{D})$.

This final result is an extension of [43, Theorem 1], with the additional characterization in terms of the Berezin transform when $\omega \in \mathcal{R}$. The condition $\frac{(\omega^\star(z))^p}{(1-|z|)^2} \in \mathcal{R}$ is not a restriction when $p \geq 1$, and equates to the condition $p > \frac{1}{2+\alpha}$ from [54, Corollary 7.17] when working with the standard Bergman weights.

5.3 SUMMARY OF PAPER III

In this article we give an atomic decomposition of the weighted mixed norm spaces $A^{p,q}_\omega$, with ω in the doubling class $\mathcal{D}$. As we saw in Proposition 5.2.6, all the functions given by a series as in (5.5) belong to the space A^p_ω, but in order to get an atomic decomposition for A^p_ω we need the reverse implication, that every function in A^p_ω could be expressed as in (5.5). This first theorem proves an analogue to Proposition 5.2.6, for the more general spaces $A^{p,q}_\omega$ with different atoms.

Theorem 5.3.1. *Let* $0 < p \leq \infty$, $0 < q < \infty$, $\omega \in \mathcal{D}$, *and* $\{z_k\}_{k=0}^\infty$ *a separated sequence in* $\mathbb{D}$. *Let* $\beta = \beta(\omega) > 0$ *and* $\gamma = \gamma(\omega) > 0$ *be those of Lemma 2.2.1(ii) and (iii). If*

$$M > 1 + \frac{1}{p} + \frac{\beta+\gamma}{q} \tag{5.6}$$

and $\lambda = \{\lambda_{j,l}\} \in \ell^{p,q}$, *then*

$$F(z) = \sum_{j,l} \lambda_{j,l} \frac{(1-|z_{j,l}|)^{M-\frac{1}{p}} \widehat{\omega}(z_{j,l})^{-\frac{1}{q}}}{\left(1-\overline{z_{j,l}}z\right)^M} \in \mathcal{H}(\mathbb{D}) \tag{5.7}$$

and there exists a constant $C = C(M, \omega, p, q) > 0$ such that

$$\|F\|_{A^{p,q}_\omega} \leq C \|\lambda\|_{\ell^{p,q}}. \tag{5.8}$$

To prove this result we use the properties of the weights in $\mathcal{D}$ given in Lemma 2.2.1 and Lemma 2.2.2, together with the results of Muckenhoupt in [35]. The next theorem proves the reverse of the previous result, that is it is proved that every function in $A^{p,q}_\omega$ can be expressed as a series of the form (5.7). In order to prove this result we shall introduce the appropriate dyadic polar rectangles induced by $K \in \mathbb{N} \setminus \{1\}$, $K > 1$.

For each $K \in \mathbb{N} \setminus \{1\}$, $j \in \mathbb{N} \cup \{0\}$ and $l = 0, 1, \ldots, K^{j+3} - 1$, the dyadic polar rectangle is defined as

$$Q_{j,l} = \left\{ z \in \mathbb{D} : r_j \leq |z| < r_{j+1}, \arg z \in \left[2\pi \frac{l}{K^{j+3}}, 2\pi \frac{l+1}{K^{j+3}} \right) \right\},$$

where $r_j = r_j(K) = 1 - K^{-j}$ as before, and its center is denoted by $\zeta_{j,l}$. For each $M \in \mathbb{N}$ and $k = 1, \ldots, M^2$, the rectangle $Q^k_{j,l}$ is defined as the result of dividing $Q_{j,l}$ into M^2 pairwise disjoint rectangles of equal Euclidean area, and the centres of these squares are denoted by $\zeta^k_{j,l}$. It is worth noticing that the cubes $Q_{j,l}$ and $Q^k_{j,l}$ as well as their centres $\zeta^k_{j,l}$ depend on the value of K.

Theorem 5.3.2. *Let $0 < p \leq \infty$, $0 < q < \infty$, $\omega \in \mathcal{D}$ and $K \in \mathbb{N} \setminus \{1\}$ such that* (2.1) *holds. Then, for each $f \in A^{p,q}_\omega$ there exists $\lambda(f) = \{\lambda(f)^k_{j,l}\} \in \ell^{p,q}$ and $M = M(p, q, \omega) > 0$ such that*

$$f(z) = \sum_{j,l,k} \lambda(f)^k_{j,l} \frac{(1 - |\zeta^k_{j,l}|^2)^{M - \frac{1}{p}} \widehat{\omega}(r_j)^{-\frac{1}{q}}}{(1 - \overline{\zeta^k_{j,l}} z)^M}, \quad z \in \mathbb{D}, \tag{5.9}$$

and

$$\left\| \{\lambda(f)^k_{j,l}\} \right\|_{\ell^{p,q}} \lesssim \|f\|_{A^{p,q}_\omega}. \tag{5.10}$$

To prove Theorem 5.3.2 some definitions and lemmas are needed. For $f \in \mathcal{H}(\mathbb{D})$, define $f_{j,l} = \sup_{z \in Q_{j,l}} |f(z)|$ and write $\lambda(f) = \{\lambda(f)_{j,l}\}$, where $\lambda(f)_{j,l} = K^{-\frac{j}{p}} \widehat{\omega}(r_j)^{\frac{1}{q}} f_{j,l}$ for all $j \in \mathbb{N} \cup \{0\}$ and $l = 0, 1, \ldots, K^{j+3} - 1$.

Lemma 5.3.3. *Let $0 < p, q < \infty$, $\omega \in \mathcal{D}$ and $K \in \mathbb{N} \setminus \{1\}$ such that* (2.1) *holds. Then $\|f\|_{A^{p,q}_\omega} \asymp \|\lambda(f)\|_{\ell^{p,q}}$ for all $f \in \mathcal{H}(\mathbb{D})$.*

This equivalent norm plays a key role in the proof of Theorem 5.3.2 as it gives us a discretization of the $A^{p,q}_\omega$-norm.

Lemma 5.3.4. *Let $0 < p \leq \infty$, $0 < q < \infty$ and $\omega \in \widehat{\mathcal{D}}$, and let $\beta = \beta(\omega) > 0$ be that of Lemma 2.2.1(ii). Then $A^{p,q}_\omega \subset A^1_\eta$ for all $\eta > \frac{\beta}{q} + \frac{1}{p} - 1$.*

In order to demonstrate Theorem 5.3.2, this lemma allows us to represent each $f \in A^{p,q}_\omega$ as $P_\eta(f)$ which help us to estimate $|f - S_\eta(f)| = |P_\eta(f) - S_\eta(f)|$, where

$$S_\eta(f)(z) = (\eta + 1) \sum_{j,l,k} f(\zeta^k_{j,l}) \frac{(1 - |\zeta^k_{j,l}|^2)^\eta}{\left(1 - \overline{\zeta^k_{j,l}} z\right)^{\eta+2}} \left| Q^k_{j,l} \right|.$$

By dividing the dyadic squares $Q_{j,l}$ in sufficiently small pieces, we can obtain $\|Id - S_\eta\|_{A^{p,q}_\omega \to A^{p,q}_\omega} \leq \frac{1}{2}$. Furthermore, we define the sequence $f_1 = S_\eta(f)$ and $f_n = S_\eta\left(f - \sum_{m=1}^{n-1} f_m\right)$ for $n \in \mathbb{N} \setminus \{1\}$ and from the estimate

$$\left\| f - \sum_{m=1}^{n} f_m \right\|_{A^{p,q}_\omega} \leq \frac{1}{2^n} \|f\|_{A^{p,q}_\omega}, \tag{5.11}$$

we reach the equality $f = \sum_{n=1}^{\infty} f_n$.

Finally we will characterize the Carleson measures for the spaces $A^{p,q}_\omega$, together with the boundedness of the differentiation operator $D^{(n)} : A^{p,q}_\omega \to L^s_\mu(\mathbb{D})$ defined by $D^{(n)}(f) = f^{(n)}$. We introduce the function

$$T_{r,u,v}(z) = \frac{\mu(\Delta(z,r))}{(1-|z|)^u \widehat{\omega}(z)^v}, \quad z \in \mathbb{D},$$

where $0 < r < 1$ and $0 < u, v < \infty$.

Theorem 5.3.5. *Let $0 < p, q, s < \infty$, $n \in \mathbb{N} \cup \{0\}$, $0 < r < 1$, μ a positive Borel measure on $\mathbb{D}$, $\omega \in \mathcal{D}$ and let $K = K(\omega) \in \mathbb{N} \setminus \{1\}$ such that* (2.1) *holds. Then the following statements are equivalent:*

(i) $D^{(n)} : A^{p,q}_\omega \to L^s_\mu(\mathbb{D})$ *is bounded;*

(ii) The sequence $\left\{ \mu(Q_{j,l}) K^{sj\left(n+\frac{1}{p}\right)} \widehat{\omega}(r_j)^{-\frac{s}{q}} \right\}_{j,l}$ *belongs to* $\ell^{\left(\frac{p}{s}\right)', \left(\frac{q}{s}\right)'}$;

(iii) The function $T_{r,u,v}$ belongs to $L_\omega^{\left(\frac{p}{s}\right)', \left(\frac{q}{s}\right)'}(\mathbb{D})$, *where*

- *(a)* $u = sn + 1$ *and* $v = 1$ *if* $s < \min\{p, q\}$;
- *(b)* $u = sn + \frac{1}{p}$ *and* $v = 1$ *if* $p \leq s < q$;
- *(c)* $u = sn + 1$ *and* $v = \frac{s}{q}$ *if* $q \leq s < p$;
- *(d)* $u = sn + \frac{1}{p}$ *and* $v = \frac{s}{q}$ *if* $s \geq \max\{p, q\}$.

Moreover,

$$\|D^{(n)}\|^s_{A^{p,q}_\omega \to L^s_\mu} \asymp \left\| \left\{ \mu(Q_{j,l}) 2^{sj\left(n+\frac{1}{p}\right)} \widehat{\omega}(r_j)^{-\frac{s}{q}} \right\}_{j,l} \right\|_{\ell^{\left(\frac{p}{s}\right)', \left(\frac{q}{s}\right)'}} \asymp \|T_{r,u,v}\|_{L_\omega^{\left(\frac{p}{s}\right)', \left(\frac{q}{s}\right)'}(\mathbb{D})}.$$

In order to prove this theorem we use Theorem 5.3.1, Rademacher functions and Khinchine's inequality (2.12) and the equivalent norm given by Theorem 5.3.3.

BIBLIOGRAPHY

[1] P. Ahern and J. Bruna, *Maximal and area integral characterizations of Hardy-Sobolev spaces in the unit ball of* $\mathbb{C}^n$, Rev. Mat. Iberoamericana **4** (1988), no. 1, 123-153.

[2] A. Aleman and J. A. Cima, *An integral operator on* H^p *and Hardy's inequality*, J. Anal. Math. **85** (2001), 157-176.

[3] A. Aleman and A. G. Siskakis, *An integral operator on* H^p, Complex Variables Theory Appl. **28** (1995), no. 2, 149-158.

[4] A. Aleman and A. G. Siskakis,*Integration operators on Bergman spaces*, Indiana Univ. Math. J. **46** (1997), no. 2, 337-356.

[5] É. Amar, *Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de* $\mathbb{C}^n$, Canad. J. Math. **30** (1978), no. 4, 711-737.

[6] A. Benedek and R. Panzone, *The space* L^p *with mixed norm*, Duke Math. J. **28** (1961), 301-324.

[7] S. Bergman, *The kernel function and conformal mapping*, Math. Surveys, **5** Amer. Math. Soc. New York, (1950).

[8] L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math.(2) **76** (1962), 547-559.

[9] L. A. Coburn, *Singular integral operators and Toeplitz operators on odd spheres*, Indiana Univ. Math J. **23** (1973), 433-439.

[10] W. S. Cohn, *Generalized area operators on Hardy spaces*, J. Math. Anal. Appl. **216** (1997), no. 1, 112-121.

[11] W. S. Cohn, and I. E. Verbitsky, *Factorization of tent spaces and Hankel operators*, J. Funct. Anal. **175** (2000), no. 2, 308-329.

[12] R. R. Coifman, Y. Meyer and E. M. Stein, *Some new functions spaces and their applications to Harmonic Analysis*, J. Funct. Anal. **62** (1985), no. 3, 304-335.

[13] R. R. Coifman, and R. Rochberg, *Representation theorems for holomorphic and harmonic functions in* L^p, Astérisque Soc. Math. Fr. **77** (1980), 11-66.

[14] O. Constantin, *Discretizations of integral operators and atomic decompositions in vector-valued weighted Bergman spaces*, Integral Equations Operator Theory **59** (2007), no. 4, 523-554.

[15] O. Constantin, *Carleson embeddings and some classes of operators on weighted Bergman spaces*, J. Math. Anal. Appl. **365** (2010), no. 2, 668-682.

[16] M. M. Dzrbashian, *Canonical representation of meromorphic functions in the unit disc*, Dokl. Akad. Nauk Arm. SSR **3** (1945), 3-9 (Russian).

[17] N. Dunford and J. T. Schwartz, *Linear operators*, John Wiley and Sons, Inc. New York, 1988.

[18] P. L. Duren, *Extension of a theorem of Carleson*, Bull. Amer. Math. Soc. **75** (1969), 143-146.

[19] P. L. Duren, *Theory of* H^p *spaces*, Pure and Applied Mathematics, Vol. 38 Academic Press, New York-London (1970).

[20] P. L. Duren and A. Schuster, *Bergman spaces*, Mathematical Surveys and Monographs, **100** American Mathematical Society, Providence, RI, (2004).

[21] F. Forelli and W.Rudin, *Projections on spaces of holomorphic functions in balls*, Indiana Univ. Math. J. **24** (1974/75), 593-602.

[22] J. B. Garnett, *Bounded analytic functions*, Pure and Applied Mathematics, **96** Academic Press, Inc. New York-London, 1981.

[23] M. Gong, Z. Lou, and Z. Wu, *Area operators from H^p spaces to L^q spaces*, Sci. China Math. **53** (2010), no. 2, 357-366.

[24] W. W. Hastings, *A Carleson measure theorem for Bergman spaces*, Proc. Amer. Math. Soc. **52** (1975), 237-241.

[25] H. Hedenmalm, B. Korenblum, and K. Zhu, *Theory of Bergman spaces*, Graduate Texts in Mathematics, **199** Springer-Verlag, New York, (2000).

[26] C. Horowitz, *Zeros of functions in the Bergman spaces*, Duke Math. J. **41** (1974), 693-710.

[27] M. Jevtić, *An embedding theorem for mixed normed spaces*, Rocky Mountain J. Math. **19** (1989), 1059-1068.

[28] N. J. Kalton, *Convexity, type and the three space problem*, Studia Math. **69** (1980/81), no. 3, 247-287.

[29] A. Khintchine, *Über dyadische Brüche*, Math. Z. **18** (1923), no. 1, 109-116.

[30] D. H. Luecking, *Trace ideal criteria for Toeplitz operators*, J. Funct. Anal. **73** (1987), no. 2, 345-368.

[31] D. H. Luecking, *Embedding derivatives of Hardy spaces into Lebesgue spaces*, Proc. London Math. Soc. **3(63)** (1991), no. 3, 595-619.

[32] D. H. Luecking, *Embedding theorems for spaces of analytic functions via Khinchine's inequality*, Michigan Math. J. **40** (1993), no. 2, 333-358.

[33] D. H. Luecking and K. Zhu, *Composition operators belonging to the Schatten ideals*, Amer. J. Math. **114** (1992), no. 5, 1127-1145.

[34] G. McDonald and C. Sundberg, *Toeplitz operators on the disc*, Indiana Univ. Math. J. **28** (1979), no. 4, 595-611.

[35] B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math. **44** (1972), 31-38.

[36] A. Nakamura, *Dual Spaces and Some Properties of $l^q(p)$, $0 < p, q \leq \infty$*, Proc. Fac. Sci. Tokai Univ. **22** (1987), 11-20.

[37] V. L. Oleinik, , and B. S. Pavlov, *Imbedding theorems for weighted classes of harmonic and analytic functions*, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **22** (1971), 94-102.

[38] J. Pau and J. Á. Peláez, *Embedding theorems and integration operators on Bergman spaces with rapidly decreasing weights*, J. Func. Anal. **259** (2010), no. 10, 2727-2756.

[39] J. Pau and R. Zhao, *Carleson Measures and Toeplitz operators for weighted Bergman spaces on the unit ball*, arXiv preprint, (2014), arXiv:1401.2563.

[40] M. Pavlović, *Mixed norm spaces of analytic and harmonic functions*, I. Publ. Inst. Math. (Beograd) (N.S.) **40(54)** (1986), 117-141.

[41] J. Á. Peláez, *Small weighted Bergman spaces*, Proc. Summer School "Complex and harmonic analysis and related topics" Mekrijärvi, June (2014).

[42] J. Á. Peláez and J. Rättyä, *Embedding theorems for Bergman spaces via harmonic analysis*, Math. Ann. **362** (2015), no. 1-2, 205-239.

[43] J. Á. Peláez and J. Rättyä, *Trace class criteria for Toeplitz and composition operators on small Bergman spaces*, Adv. Math. **293** (2016), 606-643.

[44] J. Á. Peláez and J. Rättyä, *Two weight inequality for Bergman projection*, J. Math. Pures Appl. *(9)105* (2016), no.1, 102-130.

[45] J. Á. Peláez and J. Rättyä, *On the boundedness of Bergman projection*, Advanced Courses of Mathematical Analysis VI: Proc. Sixth Int. Sch. **113** (2016).

[46] J. Á. Peláez and J. Rättyä, *Weighted Bergman spaces induced by rapidly increasing weights*, Mem. Amer. Math. Soc. **227** (2014).

[47] J. Á. Peláez and J. Rättyä, *Weighted Bergman projections on L^∞*, preprint.

[48] A. Perälä, *Vanishing Bergman kernels on the disk*, arXiv preprint,(2017), arXiv:1703.05976.

[49] F. Ricci and M. Taibleson, *Boundary values of harmonic functions in mixed norm spaces and their atomic structure*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **(4)10** (1983), no. 1, 1-54.

[50] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co. New York, (1987).

[51] Z. Wu, *Area operator on Bergman spaces*, Sci. China Ser. A **49** (2006), no. 7, 987-1008.

[52] Y. E. Zeytuncu, *Weighted Bergman projections and kernels: L^p regularity and zeros*, Proc. Amer. Math. Soc. **139** (2011), no. 6, 2105-2112.

[53] K. Zhu, *VMO, ESV, and Toeplitz operators on the Bergman space*, Trans. Amer. Math. Soc. **302** (1987), no. 2, 617-646.

[54] K. Zhu, *Operator theory in function spaces*, Monographs and Textbooks in Pure and Applied Mathematics, **139**, Marcel Dekker, Inc. New York, (1990).

[55] A. Zygmund, *Trigonometric series*, Vol. I, II. Cambridge University Press, London-New York, (1968).

UNIVERSITY OF
EASTERN FINLAND

uef.fi

**PUBLICATIONS OF
THE UNIVERSITY OF EASTERN FINLAND**
Dissertations in Forestry and Natural Sciences

ISBN 978-952-61-2718-7
ISSN 1798-5668