MIKA KOPONEN

This dissertation presents new methods of applying a Mathematical Knowledge for Teaching (MKT) framework in the context of mathematics teacher education. MKT can be used as a tool for examining developmental needs in mathematics teacher education. The present study also demonstrates an innovative way of using MKT and network analysis as tools in the investigation of teacher knowledge.
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Investigating Mathematical Knowledge for Teaching and Mathematics Teacher Education

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ABSTRACT

Deborah Ball and her colleagues at the University of Michigan have developed a practice-based theory of Mathematical Knowledge for Teaching (MKT). MKT and its measurements have been successfully applied as one of the most promising framework to describe knowledge needed for teaching mathematics in several countries. This dissertation presents new methods for applying MKT framework in the context of mathematics teacher education.

The study is divided in the dissertation into two phases. In the first phase, MKT is used as a tool for examining developmental needs in mathematics teacher education. The second phase demonstrates an innovative way of using MKT and network analysis as tools in the investigation of teacher knowledge.

In the first phase, a survey was designed and developed for investigating teacher educators’ and graduated teachers’ perceptions of the developmental needs of a mathematics teacher education program at the University of Eastern Finland. The quantitative component of the survey explored the perceptions of teachers and their educators concerning the learning experience of graduated teachers, while its qualitative component investigated the views of the same informants about the developmental needs in the contents of mathematics teacher
education. The MKT was built into both approaches to assist in clarifying the kind of developmental needs that exist in the mathematics teacher education program under scrutiny.

The findings indicate that the present mathematical contents related to Common Content Knowledge contained two challenges. University mathematics did not cover all of the topics of school mathematics, while university and school mathematics were generally only loosely connected. The findings also suggest that course contents did not take the special characteristics of the teaching profession fully into account, while school mathematics was poorly emphasized, which indicates challenges in the area of Specialized Content Knowledge.

In the teacher education program under study, the pedagogical courses were also less than adequate, and their contents did not prepare teachers to be able to identify misconceptions, learning difficulties, and other challenges that students may be confronted with in the process of learning mathematics. Furthermore, the teachers who had already graduated found that they have face problems in confronting and teaching students of varying mathematical ability. These problems were related to both the category of Knowledge of Content and Students and also that of Knowledge of Content and Teaching. In general, the graduated teachers experienced mathematical and pedagogical contents lacked in terms of both their connection with practice and also with each other. The second problem, of how mathematical and pedagogical knowledge are interconnected in the minds of teachers, persuaded us to investigate this issue in greater depth.

In the second phase, our attention focused on future mathematics teachers’ perceptions of the knowledge required for teaching mathematics, and especially to its various interrelationships. The essay data collected was first transformed into a large network and then the relationships were studied with the aid of network analysis. Strongly interconnected parts of the network, in terms of the sections of teacher knowledge, were identified using the tools of network analysis. These sections of teacher knowledge
can be used to describe not only the kind of knowledge needed in teaching but also where and why this knowledge is needed. Eleven different sections were found, but they did not match completely with the domain definitions of MKT. In addition, the sections indicated that the MKT domains are connected in a variety of ways in the minds of future teachers. Ultimately, the findings revealed that six domains are seen in a hierarchical order, which means that two domains are regarded as background knowledge, two are required for transforming background knowledge into other forms, and two are to a greater or lesser extent related to classroom actions.

Overall, the findings suggest that evaluative and development work should be a continuous and permanent process in mathematics teacher education. Recognition of the challenges of individual programs are needed so that universal challenges can be clarified. In addition, some challenges may be individual and particular to the program under scrutiny, but the information gained from recognizing such challenges is needed for further development of the mathematics teacher education program. We have been able to demonstrate that taking account of how knowledge issues are related in the minds of teachers makes research into teacher knowledge somewhat more complex, but at the same time it also renders some issues simpler. Investigating the relationships that exist in teacher knowledge remains an under-studied area of research, but we would wish to claim that we have demonstrated that network analysis can offer ways into useful exploration of this phenomenon.

*Keywords*: teacher education; mathematics teacher education; teacher knowledge; Mathematical Knowledge for Teaching; MKT; network analysis; Gephi.

*Library of Congress Subject Headings*: Mathematics teachers — Training of; Mathematics teachers — Knowledge and learning; Needs assessment; Social sciences—Network analysis; Pedagogical content knowledge.

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Mika Koponen
“Nothing is more important than to see the sources of invention which are, in my opinion, more interesting than the inventions themselves.”

- Gottfried Leibniz -
LIST OF ORIGINAL PUBLICATIONS

This thesis is based on data presented in the following articles, referred to by the Roman numerals I–III.


The above publications have been included at the end of this thesis with their copyright holders’ permission.
AUTHOR’S CONTRIBUTION

A long-term research strategy for this research work was initially discussed and decided in collaboration with the author’s research supervisors. The author has explored, selected, and established the relevant theoretical context for articles I-III. The author has also designed, collected, analyzed, and interpreted the data used in Articles I-III. In addition, the author has also undertaken a major part of the writing of articles I-III.
1 Introduction

Since Finland excelled in international assessments such as PISA and TIMSS (e.g. Andrews, Ryve, Hemmi, & Sayers 2014; Kupari, Reinikainen, & Törnroos, 2007), Finnish mathematics teacher education has received a considerable amount of attention. This success has stimulated wide international interest in Finnish education. Every year now, Finland receives visitors from around the globe and they all ask the same question: What explains this success?

There may be numerous reasons underlying the success. Based on Gallup polls, Finns rate the teaching profession as one of the most respected careers. The elementary teachers’ profession is a particularly highly desired and respected career in Finland. This may be one of the reasons why people who want to become elementary teachers are usually highly motivated. Usually, one out of ten applicants will be selected to take the elementary teacher programs in Finland (Krzywacki, Pehkonen, & Laine, 2012). Many of those who are not be selected will re-apply the following year. If teacher education programs were to be able to include qualifications tests and select their students, it is obvious that the overall level of the students selected will rise.

Finnish comprehensive school teachers are motivated, committed, and autonomous academic professionals (Krzywacki et al., 2012). They are generally well educated, since a Master’s degree is required to work as a qualified teacher (e.g. Hautamäki & Hautamäki, 2008). The Master’s degree for teachers takes about five years and includes mathematical and pedagogical courses and teaching practise. All qualified teachers have practiced teaching, and they have field experience involving school organizational and managerial issues related to school teaching. The impact of teacher education can also be witnessed at school level. Finnish students achieve better learning results in
mathematics when they are taught by qualified teachers (Hannula & Oksanen, 2013; Niemi, 2008, pp. 87-89).

According to the Finnish Ministry of Education and Culture (2010), all schools follow the guidelines detailing the national curriculum, but the freedom to arrange teaching in the most appropriate way is left to the local education authorities to decide, which may in turn be one of the reasons why Finland has succeeded so well in education. The Finnish national curriculum is somewhat unique since it has two levels. On one level, the national curriculum lays down the general principles, the teaching aims for each school level, and the minimum teaching hours per subject, while, in addition, school teachers have the responsibility of participating in making their own school-level curriculum, where they are required to make decisions about how to implement the aims of the national curriculum at the local school level (The Finnish Ministry of Education and Culture, 2013). According to Westbury, Hansén, Kansanen, and Björkvist (2005, p. 476) "As a result local school authorities, schools and teachers have been given the responsibility for the curriculum making process which in the past has been the function of the educational authorities in the national government". Since every school devises its own school-level curriculum, Finnish teachers need to know the national learning aims for each school level, but they should also have a plan for their teaching in order to be able to achieve the national aims. The freedom to arrange teaching in the most appropriate ways may be one of the reasons why Finnish teachers are both autonomous but simultaneously responsible for planning their teaching and making decisions about the curriculum (Krzywacki, 2009).

Finnish education has in general undergone major reform in the course of the past 40 years (see Hautamäki & Hautamäki, 2008). During that time, the Finnish mathematics curriculum has also undergone development. According to Kupari (1999), there have been four major changes introduced by the reform of the Finnish mathematics curriculum: The New Math, especially in 1965-1975, Back-to-Basics (1975-1985), Problem-solving (1985-2000), and Standards (1995-). Despite Finnish mathematics curriculum has
been reformed by taking account international development, each ideology also contain special characteristics of Finnish education instead of being pure replication (Kupari, 1999).

During the examined period, with the exception of Back-to-Basics, the Finnish mathematics curriculum has highlighted more and more problem solving, applied mathematical knowledge, and socio-constructivist learning process (Kupari, 1999). According to Malaty (2007), all of these curricular changes have changed Finnish school mathematics so that it will be more suitable for tests such as PISA. Krzywacki et al. (2012) suggested that in the 1980s the curriculum was a detailed document setting out the aims and contents of school subjects, but it changed in the 1990s, when a special emphasis on constructivism was written into the curriculum. At the time, problem-solving, for example, appeared in school mathematics both as a part of the content but also as a method. Krzywacki et al. (2012) have argued that nowadays the national curriculum is still reliant on the same idea of constructivism, which is why it can be argued that the main change in the reform of the Finnish mathematics curriculum occurred thirty years ago.

The Finnish success in mathematics teaching and school teaching in general can be at least largely explained by reference to the education system, the highly competent teachers, and the autonomy given to schools (The Finnish Ministry of Education and Culture, 2010). Teacher Education and Development Study in Mathematics (TEDS-M) investigated the differences between the mathematics teacher education programs in a range of countries. Some 5 000 teacher educators and 22 000 future teachers from 750 programs in some 500 teacher education institutions in 17 countries participated in the TEDS-M. Although Finland was not included in the TEDS-M study, the results contain some interesting issues that deserve discussion here.

According to the results, future teachers’ mathematical knowledge scores in the TEDS-M study correlate strongly ($R^2=0.70$, $P<.0004$) with the student achievement in TIMSS.
Interestingly, the ratio of the courses related to mathematics, mathematics pedagogy, and general pedagogy seem to play a highly significant role in mathematics teacher education. Schmidt, Houang, and Cogan (2011) discovered that when future teachers’ knowledge as measured by their TEDS-M scores was either higher or lower than, or, alternatively, in line with, the students’ achievement in TIMSS, then the detectable differences fell within the ratio of the courses in three areas. The countries where success was more frequent in the TEDS-M than in the TIMSS tended to place a greater emphasis on mathematics in their curricula. In contrast, the countries that succeeded better in the TIMSS than the TEDS-M placed a greater emphasis on general pedagogical knowledge than on mathematics. Logically, countries with greatest success in both the TEDS-M and the TIMSS placed an emphasis that lay at some point between the two general areas. Thus, it can be suggested that the extent to which these three areas are variously emphasized in teacher education seems to bear a relationship to teacher knowledge and student achievement. The results indicate that finding the balance of course-work in these three areas may be the key issue in any attempt to improve existing teacher education programs.

Previous studies also indicate that the contents of mathematics teacher education tend to have an impact on teachers’ knowledge (Schmidt, Houang, & Cogan, 2011), and teacher knowledge has an effect on how teachers teach, while the way in which teachers teach impacts on the way in which students learn (Hill, Rowan, & Ball, 2005). In addition, the contents of mathematics teacher education have an indirect effect on students’ mathematical achievements (Monk, 1994; Monk & King, 1994). This causal effect shows how crucial the contents of mathematics teacher education are for the whole education system. However, at the same time, it also contains a lot of potential for development since, if the contents of mathematics teacher education are developed, it may produce an impact on future teachers’ knowledge, and, indeed, it may also have an effect on students’ achievement. Developing the contents will also be efficient, although for it to have a broad influence, new tools for evaluating
and improving mathematics teacher education will be needed for this purpose.

Four years ago, in 2012, we decided that it was possibly an opportune moment to develop a more productive viewpoint, while at the same time no longer searching for answers to the question in the international evaluations of *what explains* students’ success, asking instead the question: *what we can do better?* In its own ways, the educational world can be as competitive as any other, and hence there is a constant need to upgrade and improve the systems that we use on a regular basis. The methods that we use to teach teachers to teach cannot remain the same from one year to another, and the circumstances in teacher education need to be evaluated critically, findings should be discussed, and teacher education should be improved. Since the content of Finnish teacher education is based on research means that teacher educators teach what they (or others) research, just as they research what they teach. What has yet to be studied is the way in which the diverse contents of Finnish mathematics teacher education work together and then also how the totality prepares teachers for their future profession. Finland was not involved in the TEDS-M study, and hence it can be stated that, for understandable reasons, fewer prior studies have been undertaken that endeavor to investigate the totality of Finnish mathematics teacher education.

### 1.1 RESEARCH PROCESS AND AIDS

The main aim of the present study has been to evaluate and improve mathematics teacher education at the University of Eastern Finland and, at the same time, to develop methods that can be used more generally to improve existing mathematics teacher education programs. Design-based research through iterative analysis offers a systematic but flexible methodology for improving educational settings (Wang & Hannafin, 2005). Iterative analysis means that assessment information is used to foster development and hence the effect of development is also
evaluated. Flexible methodology means that the research methods used can also be evaluated and developed further at every stage. The cycle can be repeated again and again, and therefore Design-based research can be seen as a sustainable strategy for enhancing development (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). In addition, Wood and Berry (2003) have argued that the characteristics of design research can be seen as an assessment tool or strategy, or alternatively as an implementation guideline for developing mathematics teacher education. All of these factors have helped us to select design-based research as the strategy for our study.

Future teachers’ and teacher educators’ perceptions are widely used as a basis for evaluating mathematics teacher education programs, and course contents, teaching coherence, the effectiveness of instruction, and also future teachers’ knowledge are important factors in measuring the effectiveness of mathematics teacher education (e.g. Hsieh et al., 2011; Tatto et al., 2008). We decided to focus on exploring teacher educators’ and graduated teachers’ perceptions about contents, teaching methods, and development needs in a whole mathematics teacher education program, and also to look in detail at how both of the respondent groups think that graduated teachers have learned various issues in the course of their teacher education.

In the course of designing the projected research work, it was unavoidable that we should consider the question of what kind of knowledge is generally needed for teaching mathematics, since this question is at least partly the same as what kind of contents teacher education should be included. The research-based understanding of the issue of what kind of knowledge is needed for teaching mathematics is already quite sophisticated, and numerous frameworks of teacher knowledge have been devised (Petrou & Goulding, 2011).

Deborah Ball and her colleagues in the University of Michigan have identified six different types of knowledge that are important in teaching mathematics. This practice-based theory is
called *Mathematical Knowledge for Teaching (MKT)* which is one of the most encouraging solutions to the question of what kind of knowledge is needed for teaching mathematics (e.g. Hill, Schilling, & Ball, 2004; Ball, Hill, & Bass, 2005; Hill, Rowan, & Ball, 2005; Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008; Ball & Bass, 2009; Ball & Forzani, 2009; Sleep, 2009). Although the basic details of MKT are located in a US context, the use of MKT has also spread in recent years to other countries, e.g. Ireland (Delaney, Ball, Hill, Schilling, & Zopf, 2008), South Korea (Kwon, Thames & Pang, 2012), Ghana (Cole, 2012), Indonesia (Ng, 2012), Norway (Fauskanger, Jakobsen, Mosvold, & Bjuland, 2012), Iceland (Jóhannsdóttir & Gísladóttir, 2014) and Malawi (Kazima, Jacobsen, & Kasoka, 2016). The growing interest of applying MKT in different countries signals the functionality of MKT. MKT can be used various ways in different context to investigate teacher knowledge. The popularity has its own benefits since each field test is a possibility for analyze and develop MKT. Applying and evaluating theoretical framework numerous times is the process for how stable theories come into the world. However, although the MKT framework seems to be similar in many respects to the aims of Finnish mathematics teacher education, how this theoretical framework works in the context of Finnish mathematics teacher education has yet to be tested.

Our research work began with the designing of a survey that could be used in collecting information about the potential development needs in mathematics teacher education (see Figure 1). The first study encapsulated teacher educators’ and graduated teachers’ perceptions of a mathematics teacher education program at the University of Eastern Finland. The survey contained four themes: content, teaching methods, the development needs of the whole program, and graduated teachers’ learning in the course of the program. The findings concerning teaching methods were published in to a nationally-circulated journal, while the data collected regarding the other three themes were published internationally. It was, however, an informed decision that persuaded us to include only the findings included in the international publications in this dissertation.
Despite this decision, in the course of the actual research process we have published three national publications that can be regarded as the general context of this dissertation, and hence they will be presented briefly in what follows.

Teacher educators’ and graduated teachers’ perceptions concerning teaching methods indicated that teaching methods do indeed require development (Koponen, Asikainen, Viholainen, &
In the mathematics teacher education studied here, a major part of the mathematics contents is taught through lectures, where students take notes. In contrast, the teaching methods used in the pedagogical studies and teaching practice normally involve group-work, discussions, and one-on-one guidance (Koponen et al., 2014). Teachers who had graduated criticized and labeled lecturing as a teacher-centered way of teaching mathematics. They claimed that if the teacher is simply lecturing while students take notes, then students are merely copying their teachers’ ideas, and hence this kind of teaching will do nothing to activate the students’ own thinking sufficiently. The same critique was also leveled at all lecture courses in the pedagogical studies.

Teacher educators who were teaching mathematics reacted to their teaching positively and could see no major problems in the way that they were teaching. Indeed, they said that they preferred demonstrative methods in teaching mathematics because it was important to illustrate mathematical ideas. In addition, they reasoned that if they were teaching a challenging area of mathematics, they tried to divide it into smaller parts so that the parts could be explicated in greater detail. Published findings indicate that some teacher educators seem to espouse a traditional approach to teaching mathematics that can be identified as instructivist in nature, whereas the approach favored by graduated teachers is more up to date and identifiable as constructivist (see Phillips, 2005).

The views of teacher educators and graduated teachers concerning the contents and their developmental needs have been published in Article I, while the views they hold about graduated teachers’ learning have appeared in Article II. These two viewpoints have constituted a consistent picture of the developmental needs for the contents of the prescribed mathematics teacher education program. Furthermore, in practice, these findings have been used to guide curriculum development work in the mathematics teacher education program currently under study. In brief, the number of courses
related to geometry, statistics, and school mathematics were increased and new courses were also developed that were designed specifically for future mathematics teachers. These changes have been reported in Koponen, Viholainen, Asikainen, and Hirvonen (2015a). However, at least one challenge still remained unanswered. After graduation, many teachers realized that they had learned mathematics and pedagogy well, but that they still had difficulty in linking these types of knowledge together and transforming them into adequately teaching mathematics. At the time, we concluded that the teacher education program had produced a somewhat fragmented picture of the knowledge required for teaching mathematics. To fix the problem, it was planned to teach a MKT framework for future teachers. We thought that the six MKT domains could possibly be arranged as a tool for broadening future teachers’ perceptions of the knowledge required for teaching mathematics in a way that finally enable them to recognize a broader and more diverse range of teacher knowledge. Our plan for this intervention appeared in Koponen, Asikainen, Viholainen, and Hirvonen (2015b). However, after reconsidering our proposals, we decided to abandon the plan. Instead of teaching a teacher knowledge framework for future teachers, we decided to investigate how future teachers regard the knowledge that is required in teaching mathematics.

In the second study, we used network analysis methods in an innovative way to investigate future teachers’ perceptions of the knowledge required for teaching mathematics and its relationships. An MKT framework was used in our analysis, and the results were published in Article III. Our findings revealed some unexpected structural features in teacher knowledge. The method that we developed for examining teacher knowledge is in fact unique and could be used in numerous ways in the future.

During the research process, we applied the MKT framework in the context of mathematics teacher education, and discovered new ways to evaluate mathematics teacher education and to investigate teacher knowledge.
1.2 RESEARCH QUESTIONS

This dissertation investigates mathematics teacher education through Mathematical Knowledge for Teaching (MKT) framework to discover relevant information for developing mathematics teacher education at the University of Eastern Finland. The dissertation seeks answers to three main research questions and eight sub-questions. The results concerning the research question A are published in Article I, research question B in Article II, and research question C in Article III. All of the sub-questions can be found in Articles I-III.

A. In which ways would teacher educators and graduated teachers develop the contents of mathematics teacher education?
   1. How do teacher educators and practicing mathematics teachers regard the course contents of mathematics teacher education?
   2. What kind of recommendations would teacher educators and practicing mathematics teachers make for improving mathematics teacher education program?

B. In which ways do teacher educators and graduated teachers decide that a graduated teacher has learned different topics?
   3. As a basis of their course contents, in which ways do teacher educators consider that graduated teachers have learned about various topics?
   4. After completing all courses of mathematics teacher education, in which ways do graduated teachers consider that they have learned about various topics?
   5. From previous appointed viewpoints, in which ways do teacher educators’ and graduated teachers’ perceptions converge?
   6. How do mathematics majors and minors consider their learning of different topics in the course of their education?

C. How do future teachers regard the knowledge required for teaching mathematics?
7. According to views expressed by future teachers, what kind of knowledge is needed for teaching mathematics?
8. How are knowledge domains related to each other in the minds of future teachers?

In what follows, attention will be paid to the kind of knowledge required for teaching mathematics by summarizing some of the prior research work in the field of teacher knowledge.
2 Theoretical background

Most people can recall their school times and identify teachers whom they considered good. Similarly, when student teachers start on a teacher education program, their ideas about good teaching are constructed from their prior experiences during their school-times (Barkatsas & Malone, 2005). Many people have ideas about what a good teacher is like, but for researchers the question is still a challenging puzzle. Many factors can be related to teachers’ professional competence, i.e. teachers’ knowledge and skills (e.g. Baumert & Kunter, 2006; Shulman, 1986), beliefs and attitudes (e.g. Ernest, 1989; Furinghetti & Pehkonen, 2002), or personality-social competence (e.g. Cisovska, 2010). Even humor can be a factor (Powell & Andresen, 1985; Torok, McMorris, & Lin, 2004; Garner, 2006). Qualification, professionalism, expertise and competence are all widely used concepts to describe the entirety of teachers' professional competence (Baumert & Kunter, 2006). However, even all of those concepts contain many elements, they cannot fully describe teachers’ professional competence. The orientation of research is always somehow delimited and therefore any theoretical concept cannot be all-encompassing. Therefore, the limited viewpoint should be selected to be the most relevant for research purposes.

There is an extensive agreement that teachers’ knowledge is a key component of teachers’ professional competence (Baumert & Kunter, 2006). Many researcher have found that mathematics teachers need strong subject knowledge and pedagogical knowledge for effective teaching (e.g. Ball, Thames, & Phelps, 2008; Ernest, 1989; Fennema & Franke, 1992; O’Meara, 2011; Rowland, Turner, Thwaites, & Huckstep, 2009). Teacher knowledge effects on how teachers teach but teaching has impact also on students’ learning (Hill, Rowan, & Ball, 2005). In addition, the course contents of mathematics teacher education have an
impact on future teachers’ knowledge but also on students’ achievement in mathematics (Hill et al., 2005; Monk, 1994; Monk & King, 1994; Schmidt, Houang, & Cogan, 2011). Furthermore, teacher knowledge gained by teachers during their teacher education might provide a key explanation for the variation in students’ test scores in mathematics internationally (see Schmidt, Houang, & Cogan, 2011).

Convincing theoretical and empirical evidences indicate that teacher knowledge is a significant perspective on mathematics teacher education and teachers’ professional competence. This is the main reason for selecting teacher knowledge as the theoretical framework for this dissertation.

2.1 TEACHER KNOWLEDGE

Teachers need knowledge about the subject that they teach and an understanding of how subject knowledge can be transformed to become understandable for students. In the 1980s the latter aspect was a missing paradigm because “no one asked how subject matter was transformed from the knowledge of the teacher into the content of instruction” (Shulman, 1986, p. 6). Shulman proposed that a teacher must have some kind of integrated knowledge of subject and pedagogy. This amalgam knowledge is needed in order to teach so that students would understand. Shulman (1986) conceptualized and named this part of teacher knowledge as Pedagogical Content Knowledge (PCK). According to Shulman’s conceptualization (1986), teachers need a combined total of three types of knowledge for effective teaching: subject matter knowledge, PCK, and curricular knowledge. Initially, PCK was considered to be a topic-specific domain that included two further subcategories: knowledge of representations and knowledge of learning difficulties, and strategies for overcoming them. Shulman’s later conceptualization contained seven categories, of which PCK was one, with no subcategories (Shulman, 1987). Shulman’s ideas and both of the
conceptualizations (1986, 1987) are fundamental and have been used as starting points by several researchers.

Many researchers have presented their frameworks of teacher knowledge in the field of mathematics (e.g. Ernest, 1989; Fennema & Franke, 1992; Baumert & Kunter, 2013; Rowland, Turner, Thwaites, & Huchstep, 2009; Ball et al., 2008; O’Meara, 2010). They have all described the different categories or domains of teacher knowledge (see Table 1).
Table 1. Main components of different teacher knowledge frameworks.

<table>
<thead>
<tr>
<th>Teacher Knowledge Framework</th>
<th>Subject matter knowledge</th>
<th>Pedagogical content knowledge</th>
<th>Knowledge of curriculum, context and education</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perspective on Teacher’s Knowledge (Shulman, 1986)</td>
<td>Subject Matter Content Knowledge</td>
<td>Pedagogical Content Knowledge</td>
<td>Curricular Knowledge</td>
</tr>
<tr>
<td>Categories of the Knowledge Base (Shulman, 1987)</td>
<td>Content Knowledge</td>
<td>Pedagogical Content Knowledge</td>
<td>General Pedagogical Knowledge</td>
</tr>
<tr>
<td>A Model of Mathematics Teacher’s Knowledge, Beliefs and Attitudes (Ernest, 1989)</td>
<td>Knowledge of mathematics</td>
<td>Knowledge of other subject matter</td>
<td>Knowledge of teaching mathematics</td>
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<td>Classroom organization and management for teaching mathematics</td>
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<tr>
<td>Components of Teacher Knowledge (Fennema &amp; Franke, 1992)</td>
<td>Content knowledge</td>
<td>Knowledge of mathematical representations</td>
<td>Pedagogical Knowledge</td>
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<td></td>
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<td>Knowledge of learning</td>
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<tr>
<td>COACTIV (Baumert &amp; Kunter 2006)</td>
<td>Content Knowledge</td>
<td>Pedagogical Content Knowledge</td>
<td>Pedagogical / Psychological Knowledge</td>
</tr>
<tr>
<td>Knowledge Quartet (Rowland et al. 2007)</td>
<td>Foundation Knowledge</td>
<td>Transformational Knowledge</td>
<td>Connection Knowledge</td>
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<td>Contingency Knowledge</td>
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<tr>
<td>Mathematical Knowledge for Teaching (e.g. Ball et al 2008)</td>
<td>Common Content Knowledge</td>
<td>Specialized Content Knowledge</td>
<td>Knowledge of Content and Teaching</td>
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<td></td>
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<td></td>
<td>Knowledge of Content and Students</td>
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<td>Knowledge of Content and Curriculum</td>
</tr>
<tr>
<td>The Ladder of Knowledge (O’Meara, 2010)</td>
<td>Subject matter Knowledge</td>
<td>Pedagogical Knowledge</td>
<td>Knowledge for Effective teaching</td>
</tr>
</tbody>
</table>
Table 1 shows that there is a broad agreement that teachers require subject matter knowledge, in this case, knowledge about mathematics. In some cases, frameworks suggest that teachers need also some kind of teaching-specific content knowledge about mathematics, which can be seen, for example, as a knowledge of mathematical representations, the history of mathematics, or the structure of mathematics (e.g. Fennema & Franke, 1992; Rowland et al. 2007; Ball et al., 2008). Furthermore, Table 1 shows that all of the researchers have held the view that teachers require pedagogical content knowledge or a type of knowledge that can be used for transferring the subject to students. Some frameworks also suggest that pedagogical content knowledge can be divided, for example, into learning and teaching separately (e.g. Shulman, 1987; Fennema & Franke, 1992; Ball et al. 2008).

Although the researchers referred to in Table 1 have suggested different categories or domains of teacher knowledge, the principal lines are relatively similar to each other and are also comparable to Shulman’s conceptualization of 1986. Petrou and Goulding (2011) suggested that in the field of mathematics many frameworks of teacher knowledge can be understand as elaborating on, rather than replacing, Shulman’s (1986) conceptualization. Petrou and Goulding (2011) proposed that the principal lines of teacher knowledge in mathematics can be synthesized by means of subject matter knowledge, pedagogical content knowledge, and curriculum knowledge in a variety of educational contexts (Figure 2). In this case, “context” should be understand as knowledge of the educational system, the aims of mathematics education, the curriculum and its associated materials (e.g. textbooks), and the assessment system (Petrou & Goulding, 2011).
The principal lines of teacher knowledge seem to be rather similar to each other if teacher knowledge is illustrated as in Petrou and Goulding’s (2011) synthesis model (Figure 2) or as in Table 1. Since researchers have their own perspective to investigate the teacher knowledge, studies generate various definitions for the domains of teacher knowledge. However, often researchers discover something common, which is why many observations can be placed under broader concepts such as Subject matter knowledge, Pedagogical content knowledge or Curriculum knowledge. However, it is possible that important details of different frameworks will be lost if their models are synthesized.

### 2.2 MATHEMATICAL KNOWLEDGE FOR TEACHING (MKT)

According to Ball and Bass (2003) the development of Mathematical Knowledge for Teaching (MKT) started with the study of empirical instructions for classroom action to identify the types of knowledge required for teaching mathematics. Documentation, lasting a year, of third grade mathematics teaching included videotapes of teaching, audiotapes of classroom lessons, transcripts, copies of students’ written classwork, homework, and quizzes, as well as the teacher’s plans, notes, and reflections (Ball et al., 2008). The analysis of the types of knowledge that teaching mathematics entailed eventually resulted in the creation of hypothetical characterizations of MKT (e.g. Ball et al., 2005; Ball et al., 2008).
The MKT framework is based on Shulman’s conceptualization (1986), but the domains are organized differently: three of the domains pertain to subject matter knowledge, while three others relate to pedagogical content knowledge (Figure 3). Within this framework, subject matter knowledge does not require pedagogical content knowledge, whereas pedagogical content knowledge requires subject matter knowledge (Ball et al., 2008).

Common Content Knowledge (CCK). Teachers need a wide knowledge of “pure mathematics”, i.e. knowledge concerning mathematical theories, concepts, definitions, results, rules, proofs, and symbols used in different areas of mathematics. This kind of knowledge can be deployed in any settings, include beyond the field of teaching, and it includes calculating, solving problems, and other common mathematics knowledge that is not unique to teaching (Ball et al., 2008). These aspects are also important in other professions such as engineering and mathematics per se, and hence they have been referred to as Common Content Knowledge (Ball et al., 2008; Hill, Ball, & Schilling, 2008).

Specialized Content Knowledge (SCK). Teachers need mathematical knowledge that is unique to teaching, for example, in selecting relevant examples, choosing appropriate exercises, evaluating
tasks, designing mathematical problems, and marking exams. Marking and grading exams are, for example, specific teaching tasks but those tasks do not require Pedagogical Content Knowledge and hence these tasks require Specialized Content Knowledge. In addition, teachers may use their knowledge of the history of mathematics or knowledge of how mathematics can be applied in their teaching, which are also examples of this specific type of mathematical knowledge (O’Meara, 2010; Jankvist, Mosvold, Fauskanger, & Jakobsen, 2015). All of the aspects that require mathematical knowledge that is unique to teaching have been termed Specialized Content Knowledge. Hill, Ball, and Schilling (2008, pp. 377-378) describe SCK as a competence that “allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems.”

Horizon Content Knowledge (HCK). Mathematics is a single, composite, accurate construction made up of mathematical concepts, and thus mathematics can be said to be concerned with concepts and with the relationships between them. In consequence, teachers must know how mathematical concepts are interrelated, how concepts form different mathematical topics, and how the structure of mathematics is constructed inside different mathematical topics. On the other hand, mathematical concepts and their definitions can be represented using different forms. Generally speaking, definitions of mathematical concepts become more exact and formal at higher levels of education. A teacher should know, for example, how the concept of function is defined at previous school levels before he/she can attach a new definition to an old one. Thus, one part of this knowledge is awareness of how mathematical concepts and their definitions can assume different forms. Ball and Bass (2003) have suggested that knowledge of this kind includes a sense of the surrounding mathematical environment, a type of knowledge they term Horizon Content Knowledge (Ball et al., 2008; Ball & Bass, 2009).
Knowledge of Content and Teaching (KCT). Teachers need to choose effective strategies for each situation and topic that they teach. Planning teaching, choosing effective strategies, arranging classrooms, promoting interaction, and communicating with their students all require pedagogical thinking. For example, if students are asking for the right answer to a given problem, a teacher can hide the answer and try to provide an opportunity for students to discover the answer by themselves. On the other hand, sometimes a student may pose a question that leads to another mathematical topic, and in those situations a teacher should make decisions about whether he/she should favor or divert the original plan. Knowledge of Content and Teaching is made up of an amalgam of knowledge of mathematics and of teaching (Ball et al., 2008).

Knowledge of Content and Students (KCS). Teachers require a knowledge of learning theories in order to understand how students learn mathematics. On the other hand, a teacher should be able to recognize how students learn mathematics in practice or the nature of the challenges that they may face. If teachers know which issues are commonly difficult for students in theory, or the nature of the misconceptions or learning difficulties that students may experience, then they will be able to prevent such problems issues and help their students to overcome challenges in practice. Furthermore, if teachers know their students and the kind of issues that they are interested in, they will be able to motivate them and make their own teaching more interesting. These aspects require knowledge of how students think or learn particular content, and hence they are generally termed Knowledge of Content and Students. Hill, Ball, and Schilling (2008, p. 375) suggest that KCS is a primary element in Shulman’s (1986) PCK, since one part of Shulman’s PCK is “an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons”.
Knowledge of Content and Curriculum (KCC). The national curriculum normally presents the guidelines and aims for the mathematics teacher profession, i.e., concerning the kind of mathematical topics that should be taught and the nature of the other aims that are concerned with school teaching at the various levels. Teachers need to be aware of the various guidelines and aims involved in the teacher profession. In addition, teachers should have a knowledge of the range and use of teaching materials (such as textbooks, other materials, etc.), teaching instruments (blackboards, overhead projectors, etc.), and technology (computers, smart boards, calculators, software, etc.). The use of materials, instruments, or technology in teaching requires integrated knowledge of subject matter knowledge, pedagogical content knowledge, and knowledge about equipment (Koehler & Mishra, 2009). All of these aspects of knowledge can be summed up in terms of Knowledge of Content and Curriculum (Ball et al., 2008; Jankvist et al., 2015).

After the hypothetical characterization of MKT was created, a number of specific measurements of MKT were developed to test this kind of knowledge (Hill, Schilling, & Ball, 2004). Thus, these measurements were tested against the practical reality (Hill, Blunk, Charalambous, et al., 2008) and also against students’ actual achievements (Hill, Rowan, & Ball, 2005). Thereafter, these measurements have been translated into other languages and have been applied in other countries, e.g. Ireland (Delaney et al., 2008), South Korea (Kwon et al., 2012), Ghana (Cole, 2012), Indonesia (Ng, 2012), Norway (Fauskanger et al., 2012), Iceland (Jóhannsdóttir & Gísladóttir, 2014), and Malawi (Kazima et al., 2016). There has also been a growing interest in applying this framework within the Nordic context (e.g. Mosvold, Bjuland, Fauskanger, & Jakobsen, 2011; Fauskanger et al., 2012; Mosvold & Fauskanger, 2013; Jóhannsdóttir & Gísladóttir, 2014; Jankvist et al., 2015).
2.3 THE MAINLINES IN THE RESEARCH OF TEACHER KNOWLEDGE

Internationally, there exists a broad research interest in teacher knowledge. A review study by Hoover, Mosvold, Ball, and Lai (2016) indicates that teachers’ knowledge has already been measured in several regions (e.g. Africa, America, Asia, Europe, and Oceania) and at all levels of school (e.g. primary, middle, secondary, tertiary). However, a majority of the studies were carried out in North America and focused on measuring primary teachers’ knowledge (Hoover et al., 2016). Interestingly, those studies where a standardized instrument was used to measure teachers’ knowledge, the instrument was in most cases based on the MKT framework (Hoover et al., 2016). The result indicates that the MKT measures are popular and widely used.

Analysis of the research problems revealed that most of studies reviewed focused on developing teachers’ knowledge in support of teaching, while another two mainstreams examined the nature and the impact of teacher knowledge (Hoover et al., 2016). An earlier study also offered a systematic review of the way in which Shulman’s (1986) original PCK has been conceptualized and studied empirically in mathematics education research (see Depaepe, Verschaffel, & Kelchtermans, 2013). Depaepe et al. (2013) found six distinct mainstreams in research focusing on teacher knowledge. Similarly, in contrast to Hoover et al. (2016), two principal lines examined the nature and the development of PCK. Four other principal lines explored the relationship between PCK and 1) Content Knowledge, 2) instructional practice, 3) students’ learning outcomes, and 4) personal characteristics. These last four principal lines are to a greater or less extent concerned with the impact of teacher knowledge.

We consider that both review studies predict similar results. In terms of research into teacher knowledge, investigating how teachers’ knowledge can be developed is large mainstream, and both the nature of teacher knowledge and the impact of teacher knowledge are less emphasized, but undoubtedly part of mainstream as well (see Depaepe et al., 2013; Hoover et al., 2016).
Interestingly, these three mainstreams in teacher knowledge research are quite similar to each other compared with Shulman’s concerns of thirty years previously.

2.3.1 A review of Shulman’s concerns in 1986

Shulman’s framework encapsulating teacher knowledge has undoubtedly guided research into teacher knowledge, and hence it is relevant to look back to his concerns as he expressed them in 1986. Shulman (1986) argued that, when investigating teacher knowledge, it is important to focus attention on how teachers themselves view the knowledge required for teaching. Furthermore, it is important, he claimed, to examine the kind of domains and categories that teacher knowledge involves, the ways in which the domains are interrelated, and the best ways to develop this teacher knowledge. Shulman (1986) suggested that the following questions were central to research into teacher knowledge:

- What are the domains and categories of content knowledge in the minds of teachers?
- How, for example, are content knowledge and general pedagogical knowledge related?
- In which forms are the domains and categories of knowledge represented in the minds of teachers?
- What are promising ways of enhancing acquisition and development of such knowledge? (p. 9)

Those concerns are all related to the process of gaining an understanding of how teachers’ knowledge can be developed and also of the nature and impact of teacher knowledge. Hence, we would claim that Shulman’s concerns of three decades ago are rather similar to the main lines of present-day research into teacher knowledge.

In the context of MKT, the relationships between the six MKT domains remain less studied. Ball et al. (2008) noted that many of the demands imposed by teaching mathematics require a
knowledge of the intersection of the six MKT domains. They illustrate this problem through the following example: “In other words, recognizing a wrong answer is common content knowledge (CCK), whereas sizing up the nature of an error, especially an unfamiliar error, typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of specialized content knowledge (SCK). In contrast, familiarity with common errors and deciding which of several errors students are most likely to make are examples of knowledge of content and students (KCS)” (p. 401). In light of this perspective, it can be claimed that the domains of MKT are interrelated in a variety of ways, but that a problem still resides on Shulman’s (1986) notion that the relations between the domains of teacher knowledge requires examination.

Many frameworks of teacher knowledge respond to the question of what kind of knowledge is required for teaching mathematics, but they rarely respond to the question of how the domains or categories of knowledge are related to each other. Baumert and Kunter (2013, p. 28) suggested that “there is far less agreement about the structure of this [teacher] knowledge, the different types of knowledge and their epistemological status, or the development and mental representation of professional knowledge and skills”. Hashweh (2005) also argued that Shulman neglected the potential for integration between these categories and the hierarchies that may exist between them, but he left the task of further development of the concept to other researchers.
3 Methodology

In this chapter we will start by providing an overview of the research paradigm and strategy that underlie this study. Subsequently, we will turn our attention to our research design, instruments, context, sample, and data analysis methods.

3.1 RESEARCH PARADIGM AND STRATEGY

A research paradigm can be characterized in terms of ontological, epistemological, methodological, axiological, and rhetorical beliefs (e.g. Guba & Lincoln, 1994; Gunnarsson, 1998; Onwuegbuzie & Johnson, 2006). Thus, the research paradigm describes fundamentally how scientific knowledge has been developed. According to Guba and Lincoln (1994), questions such as "what is the form and nature of reality and, therefore, what is there that can be known about it?" are ontological questions; "what is the nature of relationships between the knower or would-be knower and what can be known?" are epistemological questions; and "how can the inquirer (would-be knower) go about finding out whatever he or she believes can be known?" is a methodological question (p. 108).

Ontological, epistemological and methodological questions are unavoidably interconnected, but also in a hierarchical order. The answers to methodological questions depend on epistemological beliefs, while the answers to epistemological questions depend on ontological beliefs (Guba & Lincoln, 1994). Researchers’ own values (axiological beliefs) and the use of informal or formal writing styles (rhetorical beliefs) are also aspects of the research paradigm (Onwuegbuzie & Johnson, 2006). According to Onwuegbuzie and Johnson (2006), all five of these aspects of the research paradigm should be considered both at the start but also during the final evaluation of a research project (Figure 4).
Figure 4. Aspects of the research paradigm.

The paradigm chosen for the present research work can be placed under pragmatism. Pragmatism emphasizes the research problem, but it cannot be placed under any other specific philosophy (Creswell, 2009; Biesta, 2010; Greene & Hall, 2010). Pragmatism should be understood as a set of philosophical tools that can be used to address problems (Creswell, 2009; Biesta, 2009; 2010). Ontologically, “pragmatists do not see the world as an absolute unity” (Creswell, 2009, p. 11). As a result of this worldview, located under pragmatism, none of the methodological approaches is fundamentally better than any of the others (Creswell, 2009; Biesta, 2010). In consequence, in methodological terms pragmatism allows us to claim that there is no single approach, but that all approaches may be accepted (Biesta, 2010). All qualitative, quantitative, or mixed techniques are accepted if they increase understanding of the research problem.

In Greene’s (2008) view, because of this epistemological and methodological flexibility of pragmatism, the popularity of mixing qualitative and quantitative approaches is unsurprising. Creswell (2009, p. 11) also considered that “for the mixed methods researcher, pragmatism opens the door to multiple methods, different worldviews, and different assumptions, as well as different forms of data collection and analysis”. As a result, pragmatism may provide a philosophy that supports paradigm integration and mixing.
quantitative and qualitative techniques (Johnson, Onwuegbuzie, & Turner, 2007).

In our opinion, researchers need to think seriously about their research paradigm, but they should also consider the long-term concerns of the research. Researchers’ eyes should not only be directed toward the immediate problems in their research but also toward the more distant future. In other words, in addition to the researcher needing to focus on current research problems, he/she should develop a long-term plan for their research work. Hence, long-term planning and designing could well be termed a research strategy.

According to Wang and Hannafin (2005) “we define design-based research as a systematic but flexible methodology aimed to improve educational practices through iterative analysis, design, development, and implementation, based on collaboration among researchers and practitioners in real-world settings, and leading to contextually-sensitive design principles and theories” (pp. 6-7). A fundamental principle for Design-Based Research is to repeat a process of assessment and development (e.g. Edelson, 2002; Richey, Klein, & Nelson, 2004; Design-Based Research Collective, 2003; Collins, Joseph, & Bielaczyc, 2004). Wood and Berry (2003, p. 197) suggested that developing mathematics teacher education is the same as “flying an airplane and trying to fix it at the same time”. In consequence, they have claimed that the dynamic and iterative nature of design based-research can offer an excellent strategy for developing mathematics teacher education. Cobb et al. (2003) have also suggested that Design-Based Research works well in the evaluation and improvement of educational settings.

Pragmatism as a research paradigm, design-based research as research strategy, and mixed methods research as research techniques were all reasonable and compatible choices for our research intentions, purposes, and worldview, and these choices have continued to guide the choices that we have made regarding our research work.
The first short-term aim of this research was to collect information about the developmental needs of the mathematics teacher education program at the University of Eastern Finland, while its long-term aim was to develop general methods that can be used in evaluating, and possibly also improving, mathematics teacher education programs. Our research started out by investigating potential challenges and developmental needs in mathematics teacher education, but we found that, there existed no relevant instrument for this purpose. In consequence, we decided to develop our own instrument.

One way to evaluate the effectiveness of mathematics teacher education is by investigating the perceptions of teacher educators and student teachers (Hsieh et al., 2011). However, student teachers’ perceptions may be limited to some extent, since they have no practical experience of working as school teachers. We concluded, therefore, that examining the perceptions of graduated teachers might be more effective. We decided to investigate teacher educators’ and graduated teachers’ perceptions of content, teaching methods, and their developmental needs, and also their perceptions of graduated teachers’ own learning. Survey research is an effective technique for examining respondents’ perceptions, but in the present case the potential population sizes may differ considerably, since the number of graduated teachers is large compared with teacher educators. As a result, we then started to think that perhaps neither qualitative or quantitative data alone would be able to provide a full understanding of our research problems, and therefore we chose to collect both qualitative and quantitative data simultaneously, which is a recognized configuration used in mixed-methods research (Creswell & Clark, 2011). In the present case, qualitative and quantitative data types may supplement each other, and integrating the findings of these data types may in turn deepen and broaden our understanding of the phenomena under examination (Johnson et al., 2007; Tashakkori & Creswell, 2007). In addition, this decision to use mixed methods was taken for practical reasons. If we were unable to contact a large number of respondents, which may sometimes
occur in survey studies, then we would still be able to place more focus on qualitative data. On the other hand, if we do have a large number of respondents then we can compare quantitative- and qualitative-based results. Only the latter version can, however, be identified in terms of mixed methods research, since integrating both data types plays a significant role in mixed methods research. Integration means that qualitative and quantitative data needs to be integrated by a previously undetermined method during the research process (e.g. data collection, analysis, interpreting results, conclusions). In the case of this study, both data types were collected simultaneously, but they were treated separately. The qualitative findings appeared in Article I and the quantitative findings in Article II. Their integration will occur in this dissertation, which is why the results will be discussed simultaneously in the result section. These findings offered new knowledge about the mathematics teacher education program under examination, and hence it is reasonable to argue that, up to this point at least, we have followed the principles of design-based research. Epistemologically, design-based research begins with an analysis of practical problems (see Figure 5).

![Design-based research](image)

*Figure 5. Epistemologically design-based research begins with analysis of practical problems (Amiel & Reeves, 2008).*

In the case of the present project, the strategy chosen revealed, for example, that some graduated teachers had learned mathematics and pedagogy but still experienced difficulty in integrating these two knowledge types and deploying them in their mathematics teaching. Initially, we were unable to find an appropriate response to this challenge and therefore we decided that the problem required further research attention. At the same time, we re-evaluated our previous methods of analysis. We found in the
process that there was a possibility of improving the method of analysis that we had used for analyzing open questions. In consequence, in the subsequent study we integrated network analysis methods into the previous method of analysis. We now consider that this constitutes a legitimate way of combining pragmatism and the strategy of design-based research in such a way that it can offer support even in “flying and fixing the airplane at the same time”. The selected strategy persuaded us to rethink our previous choices and thus develop the research methods used here. In the present case, therefore, rethinking played an important role in the further development of the methods of analysis. We also think that this research work can clearly be located under pragmatism, and the chosen research strategy is undoubtedly suitable for our present purposes and also in terms of the characteristics of pragmatism.

3.2 RESEARCH DESIGN

Our research process can be divided into two main phases (see Table 2). In the first phase, we made use of surveys to examine 1) how graduated teachers and teacher educators viewed the course contents and their developmental needs in mathematics teacher education, and furthermore 2) how both respondent groups felt that graduated teachers had learned different topics in the course of their education. To aid investigation of the first research question, the survey included open questions related to the course contents and their developmental needs in mathematical, pedagogical studies, and during teaching practice. To advance the investigation of the second research question, the survey included 72 statements related to the knowledge required for teaching mathematics. Because the data types differed, we treated and analyzed them separately. This, in turn, justified separate publication of the results based on open question in Article I and those based on statements in Article II.
Table 2. Main components of the research design.

<table>
<thead>
<tr>
<th></th>
<th>First research phase</th>
<th>Second research phase</th>
</tr>
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<tbody>
<tr>
<td>Research question</td>
<td>In which ways would teacher educators and graduated teachers develop the contents of mathematics teacher education?</td>
<td>In which ways do teacher educators and graduated teachers consider that a graduated teacher has learned different topics?</td>
</tr>
<tr>
<td>Instrument</td>
<td>Survey</td>
<td>Essays</td>
</tr>
<tr>
<td>Data type</td>
<td>Open questions (Qualitative)</td>
<td>72 items (Quantitative)</td>
</tr>
<tr>
<td>Analysis</td>
<td>Theory-driven content analysis</td>
<td>Quantitative methods</td>
</tr>
<tr>
<td>Results</td>
<td>Article I</td>
<td>Article II</td>
</tr>
</tbody>
</table>

In the second phase of the research, we looked at the ways in which future teachers regard the knowledge required in the teaching of mathematics. The student teachers wrote an essay under the rubric of the knowledge required for teaching mathematics. The data was analyzed using a method similar to that described in Article I, but on this occasion we also focused on how the knowledge topics referred to were related to each other. Hence, for this purpose we used network analysis methods. The results appeared in Article III.

3.3 INSTRUMENTS

In the first phase of the research, data was collected by means of a survey. The survey contained open questions and seventy-two statements about the knowledge required in the teaching of mathematics. In the second phase of the research, essays were used as the instrument for collecting data.
3.3.1 Survey

According to Fowler (2013), survey research integrates sampling, question design, and data collection methodologies, and hence there are several aspects to take into consideration in the collecting, analyzing, and reading about the survey data. The data could also be read differently. In survey studies there are many points at which mistakes can be made (Fowler, 2013). For example, in the US context the best-known surveys are related to the outcome of election (Krosnick, 1990). In the early twentieth century surveys predicted results reasonably well, but in 1948 Truman won, rather than Dewey, whose election had been predicted in the survey (Krosnick, 1990). According to Krosnick (1990), the nonsystematic methods used at the time were expected to generate credible samples of respondents, which in that famous case produced a sample that was unrepresentative of the general population, eventually triggering the fault result. An example such as this demonstrates precisely why survey studies need to be designed with considerable care.

No ready-made survey existed to suit our purposes, and therefore we needed to design one of our own. The resultant survey eventually contained 72 statements related to the knowledge required in the teaching of mathematics and also open-ended questions related to the course contents and their development needs in mathematics teacher education.

Questions that can be misunderstood constitute a common fault in survey studies, and pre-testing or a piloting survey are recommended (Krosnick, 1990; Fowler, 2013). First, in the case of pre-testing, four independent researchers improved the language and intelligibility of all of the open questions and statements. Secondly, in support of piloting, four volunteers participated in pre-testing the survey. Two of the volunteers were from the study group and the other two came from outside the study group. It was thought that individuals from outside the study group might be better at evaluating the intelligibility of the wording and also in commenting on the abstractness of the concepts used. All of
the eight people who participated in the pre-testing and piloting expressed their opinions about the intelligibility of the instructions, statements, and questions, and also commented on how they would develop the survey. On the basis of the pre-testing, only a few words were replaced because of their perceived ambiguity.

Two versions of the survey were made, one intended for the teacher educators and the other for the graduated teachers. The content of both surveys was similar apart from minor differences in their apparent perspectives.

The survey devised for the graduated teachers contained three open questions.

1. Evaluate the contents of mathematical studies in the mathematics teacher education program, with particular regard to the work of a mathematics teacher.
2. Evaluate the contents of the pedagogical studies and teaching practice in the mathematics teacher education program, with particular regard to the work of a mathematics teacher.
3. Suggestions regarding the development of mathematics teacher education would also be appreciated.

The survey for teacher educators included two open questions.

4. Evaluate the contents of the studies you teach, with particular regard to the work of a mathematics teacher.
5. It would also be appreciated if you could provide suggestions regarding the development of mathematics teacher education.

Both the teacher educators’ and the graduated teachers’ surveys included the same seventy-two statements concerning the knowledge related to teaching mathematics. The question presented to the graduated teachers was in the form: “How would you rate the knowledge / skills that your mathematics teacher education has provided you with?”. And for the teacher educators: “Based on your course contents, how would you rate the knowledge / skills that
both groups of respondents used the five-point Likert scale (1 = not at all, 2 = poor, 3 = fair, 4 = good, 5 = excellent).

The statements were based on the literature, definitions, and descriptions of the six MKT domains (e.g., Ball, 2003; Hill, Schilling, & Ball, 2004; Hill, Rowan, & Ball, 2005; Ball et al., 2008; Ball & Hill, 2008; Sleep, 2009; Ball & Forzani, 2009). For the sake of accuracy, the statements were based on our own understanding at that time of the six MKT domains. In consequence, the following descriptions are presented here in exactly the same form as they appear in Article II.

Common Content Knowledge (see Table 3). A teacher requires, more than anything else, a knowledge of a range of different mathematical topics. In Finland, for example, the mathematical topics are prescribed in the national curriculum for each grade (e.g., Finnish National Board of Education, 2003). Twelve statements outline the mathematical topics, which cover roughly all of the necessary topics at secondary school level in the country (Finnish National Board of Education, 2003). The topics are concerned with calculus (2), functions (4), geometry (2), data and probability (2), and numbers and vectors (2). Because Common Content Knowledge is actually more extensive than a knowledge of the prescribed topics, some of the items are also connected with mathematical concepts, notations (2), and methods (4). Two of the statements are also connected with study skills, since both student teachers and teachers need to possess the requisite skills for solving mathematical problems and also the ability to read mathematical texts (2).

Table 3. Themes and items concerned with Common Content Knowledge.

<table>
<thead>
<tr>
<th>Themes</th>
<th>Content of items</th>
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<tbody>
<tr>
<td>Calculus</td>
<td>Differential calculus</td>
</tr>
<tr>
<td></td>
<td>Integral calculus</td>
</tr>
<tr>
<td>Functions</td>
<td>Functions and equations</td>
</tr>
<tr>
<td></td>
<td>Polynomial functions</td>
</tr>
<tr>
<td></td>
<td>Root and logarithmic functions</td>
</tr>
<tr>
<td></td>
<td>Trigonometric functions</td>
</tr>
</tbody>
</table>
Geometry | Analytic geometry
---|---
Geometry | Geometry
Data and probability | Probability theory
Statistics | Statistics
Numbers and vectors | Number sequences
Vector calculus | Vector calculus
Mathematical concepts and notations | Exact use of mathematical notations
Mathematical concepts | Mathematical concepts
Studying skills | Solving mathematical exercises and problems
Reading and understanding university-level course materials used in mathematical studies | Reading and understanding university-level course materials used in mathematical studies
Mathematical methods | Mathematical calculation methods
Mathematical reasoning rules (e.g., logic) | Mathematical reasoning rules (e.g., logic)
Use of suitable mathematical methods | Use of suitable mathematical methods
Use of figures, diagrams, and models to promote own mathematical thinking | Use of figures, diagrams, and models to promote own mathematical thinking

Specialized Content Knowledge (see Table 4). In the classroom, teachers present mathematics in a variety of ways, and this process requires a knowledge of mathematical representations and strategies useful in visualizing mathematics (3). Teachers also need to have skills that enable them to formulate new assignments and exams, and to evaluate and grade their students’ output (3). Sometimes a teacher in the classroom will attempt to justify the learning of mathematics, and hence it may well be advantageous for the teacher if he or she knows something about the history and philosophy of mathematics (2) or how mathematics has influenced everyday life, culture, and society (2).

Table 4. Themes and items concerned with Specialized Content Knowledge.

<table>
<thead>
<tr>
<th>Themes</th>
<th>Content of items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Presenting mathematics</td>
<td>Mathematical representations (verbal, picture, symbolic)</td>
</tr>
<tr>
<td></td>
<td>Use of multiple representations (verbal, picture, symbolic) of the same mathematical entity</td>
</tr>
<tr>
<td></td>
<td>Use of visualizations in mathematics</td>
</tr>
<tr>
<td>Formulation and marking assignments</td>
<td>Formulation of mathematical exams</td>
</tr>
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<td></td>
<td>Formulation of mathematical exercises</td>
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<tr>
<td></td>
<td>Marking exam responses</td>
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<tr>
<td></td>
<td>Recognition of straightforward and more problematic mathematical exercises</td>
</tr>
<tr>
<td>Influence of mathematics</td>
<td>Mathematical applications in everyday life, science, and technology</td>
</tr>
</tbody>
</table>
Horizon Content Knowledge (see Table 5). In very general terms, mathematics can be said to be concerned with concepts and with the relationships between them. In consequence, it might be said that mathematics is ultimately a single accurate construction made up of mathematical concepts. In this light it is clear that a teacher needs to possess knowledge of mathematical structures (3). However, experts and novices may well see the structure of mathematics differently, and therefore teachers need to know how the structure of mathematics can be seen differently before teachers can intervene to support the understanding of mathematical structures (3).

### Table 5. Themes and items concerned with Horizon Content Knowledge.

<table>
<thead>
<tr>
<th>Themes</th>
<th>Content of items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical structures</td>
<td>Hierarchy of mathematical concepts (e.g. axioms, definitions, lemmas, propositions)</td>
</tr>
<tr>
<td></td>
<td>Fields of mathematics in general</td>
</tr>
<tr>
<td></td>
<td>Structure of concepts in different mathematical fields (e.g. how mathematical concept of limit is related to other concepts)</td>
</tr>
<tr>
<td>Learning of mathematical structures</td>
<td>Teaching the structure of mathematics</td>
</tr>
<tr>
<td></td>
<td>Recognition of previous and forthcoming mathematical concepts in teaching mathematics</td>
</tr>
<tr>
<td></td>
<td>Students’ actual mathematical know-how at different school levels (e.g., typical issues in the need for revision with students)</td>
</tr>
</tbody>
</table>

Knowledge of Content and Teaching (see Table 6). A teacher needs knowledge and skills concerned with the planning of his/her teaching (2). To become effective in his/her teaching, a teacher should probably take into account how students generally learn mathematics. Thus, teachers’ professional skills may also develop if they apply their knowledge of learning theories to their own teaching (2). In the classroom teachers need access to various skills connected, for example, with discussing mathematics (3).
and using examples (2). To achieve the aims of their teaching, teachers may also need to underline the most significant issues and explain the aims of the teaching (3). In general, an effective teacher will endeavor to generate a positive experience of the general process of learning mathematics (2).

Table 6. Themes and items concerned with Knowledge of Content and Teaching.

<table>
<thead>
<tr>
<th>Themes</th>
<th>Content of items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Planning of teaching</td>
<td>Planning and defining the learning objectives of individual lessons</td>
</tr>
<tr>
<td>Planning of teaching</td>
<td>Planning and defining the learning objectives of a course or unit of study</td>
</tr>
<tr>
<td>Applying learning theories to teaching</td>
<td>Constructivism and its application to one’s own teaching</td>
</tr>
<tr>
<td>Applying learning theories to teaching</td>
<td>Learning theories and their application in the teaching</td>
</tr>
<tr>
<td>Applying learning theories to teaching</td>
<td>Research results of unqualified mathematics and their application in their teaching</td>
</tr>
<tr>
<td>Explaining aims of teaching</td>
<td>Underlining the most important aspects of the topic to be learned</td>
</tr>
<tr>
<td>Explaining aims of teaching</td>
<td>Explaining to students the aims and meanings of the topic under study</td>
</tr>
<tr>
<td>Explaining aims of teaching</td>
<td>Underlining the significant aspects of mathematical content</td>
</tr>
<tr>
<td>Producing learning experiences</td>
<td>Generation of positive experiences in the process of learning mathematics</td>
</tr>
<tr>
<td>Producing learning experiences</td>
<td>Teaching self-evaluation skills to students</td>
</tr>
<tr>
<td>Discussion of mathematics</td>
<td>Presenting specific questions to promote learning</td>
</tr>
<tr>
<td>Discussion of mathematics</td>
<td>Talking about mathematics</td>
</tr>
<tr>
<td>Discussion of mathematics</td>
<td>Answering students why-questions in mathematics</td>
</tr>
<tr>
<td>Use of examples</td>
<td>Use of everyday life examples in teaching mathematics</td>
</tr>
<tr>
<td>Use of examples</td>
<td>Formulation of relevant examples for teaching specific topics</td>
</tr>
</tbody>
</table>

Knowledge of Content and Students (see Table 7). A teacher needs to evaluate students’ behavior, and their thinking and speaking at the level of the individual (5). A teacher may be able to identify students’ misconceptions of mathematics and also prevent them, if the teacher him-or herself possesses some knowledge of the challenges posed by the learning of mathematics (4). A teacher needs to know about supporting and motivating students in their learning (2), because ultimately the teacher is attempting to improve students’ mathematical competence (3).
Table 7. Themes and items concerned with Knowledge of Content and Students.

<table>
<thead>
<tr>
<th>Themes</th>
<th>Content of items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Improving students’ competence</td>
<td>Developing students’ problem-solving skills</td>
</tr>
<tr>
<td></td>
<td>Developing students’ mathematical thinking</td>
</tr>
<tr>
<td>Evaluating students’ competence</td>
<td>Evaluation of students’ competence</td>
</tr>
<tr>
<td></td>
<td>Students’ problem-solving and mathematical reasoning skills demonstrating their deep understanding</td>
</tr>
<tr>
<td></td>
<td>Evaluation of the accuracy and coherence of students’ conclusions</td>
</tr>
<tr>
<td></td>
<td>Recognition of students’ meaningful talk in their learning of mathematics</td>
</tr>
<tr>
<td></td>
<td>Assessment of students’ mathematical knowledge, skills, and ability for their future studies</td>
</tr>
<tr>
<td>Challenges of learning</td>
<td>Recognition of students’ errors and taking them into account in teaching</td>
</tr>
<tr>
<td></td>
<td>Prevention of students’ misconceptions</td>
</tr>
<tr>
<td></td>
<td>Recognition of students’ misconceptions in mathematics (e.g. multiplication increases and division decreases results)</td>
</tr>
<tr>
<td></td>
<td>Recognition of learning difficulties in mathematics</td>
</tr>
<tr>
<td>Motivation and support skills of learning</td>
<td>Motivating students to learn, understand, and know mathematics</td>
</tr>
<tr>
<td></td>
<td>Supporting mixed-level students in learning mathematics</td>
</tr>
<tr>
<td></td>
<td>Supporting students’ self-confidence in their mathematical skills</td>
</tr>
</tbody>
</table>

Knowledge of Content and Curriculum (see Table 8). In most countries, curricula are regarded as defining specific learning objectives, and hence curricular knowledge will also provide relevant knowledge for a teacher (2). Technology, textbooks, and other equipment will be useful tools for both learning and teaching. Teachers will require this variety of knowledge when selecting and using textbooks in their teaching of mathematics (2). The same will be true of knowledge and skills concerned with the use of technology in teaching mathematics (2).

Table 8. Themes and items concerned with Knowledge of Content and Curriculum.

<table>
<thead>
<tr>
<th>Themes</th>
<th>Content of items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Use of textbooks</td>
<td>Use of textbooks in teaching mathematics</td>
</tr>
<tr>
<td></td>
<td>Choosing a suitable textbook or textbook series suited to one’s own teaching style</td>
</tr>
<tr>
<td>Curricular knowledge</td>
<td>Objectives and contents of the national mathematical curriculum</td>
</tr>
</tbody>
</table>
The total number of statements was designed to ensure that half were connected with subject matter knowledge and half with pedagogical content knowledge in terms of MKT.

### 3.3.2 Essays

In the second phase of the research we examined student teachers’ perceptions regarding the knowledge required for teaching mathematics. The student teachers wrote an essay under the rubric of “The kind of knowledge required for teaching mathematics”. Data was collected during a course devoted to *Analysis skills for teaching mathematics*. The course was intended for student teachers approaching the end of their education. To write the essays the student teachers used laptops with no internet connection, and they were given a total of two hours to complete the task. No minimum or maximum limits were imposed on the eventual length of the essays.

It was decided to use the essay format as the mode of data collection because we hoped that writing a story would persuade student teachers not only to describe the knowledge required for teaching but also to provide explanations, reasons or justifications for why they had written about certain aspects and topics rather than writing, for example, simply a list of knowledge issues. We hoped by this means that we would be able to analyze various aspects of knowledge already mentioned but also their relations or contexts as they appear in the text data.

There are several ways in which this kind of text analysis can be done. For example, *Latent Semantic Analysis (LSA)* is a fully automatic text data analysis method that transforms each word and its text passage into a matrix, and then with the aid of factor analysis it examines the relationships between words and their contexts (Landauer, Foltz, & Laham, 1998). LSA is fully automatic,
no human-constructed dictionaries are used, and there is no need to undertake interpretations (Landauer & Dumais, 1997). *Linguistic Inquiry and Word Count (LIWC)*, on the other hand, counts words, compares written words with dictionaries, analyses word contexts, and detects meaningful categories for words (Tausczik & Pennebaker, 2010).

The network analysis methods can be also used in text analysis. For example, Gephi, which we used here, is an open-source software resource designed for exploring and manipulating a network (Bastian, Heymann, & Jacomy, 2009). Several examples exist that demonstrate how Gephi can be used for in analyzing text and illustrating text data (e.g. Song & Kim, 2013; Stieglitz & Dang-Xuan, 2013; Myneni, Cobb, & Cohen, 2013). In the case of text analysis, Gephi can be used, for example, to identify the text’s main agenda, grouping words with similar contexts, and exploring the plot structure of a narrative story (Paranyushkin, 2011). Gephi is not, however, fully automatic text analysis software, which means that the text data must be transformed into nodes and their relationships. This kind of text analysis is usually known as *content analysis*. We used two standard approaches, Conventional content analysis for placing certain aspects of knowledge in data-based categories, and Direct content analysis, which can locate particular aspects of knowledge and place them in theory-based categories (Hsieh & Shannon, 2005).

In our opinion, however, a coding process by means of which certain aspects of knowledge (whole sentences and their meanings) can be assigned to data-based and theory-based categories cannot be performed by machine. In consequence, we made no use of any fully automated text analysis software. The coding process allowed us to place the same topics in the same category. If same topic happens to be written in very different modes of language it is likely that automatic text analysis software will be unable to put them in the same category. At the same time, we traced the ways in which topics concerning knowledge are interrelated, something that is also likely to be challenging for any automatic text analysis software (Figure 6).
3.4 CONTEXT

All of the data collection was conducted under the auspices of the University of Eastern Finland. The University of Eastern Finland offers three routes for students majoring in mathematics and two routes for students taking mathematics as a minor (Table 9).

Table 9. Course work for various MS degrees at the University of Eastern Finland.

<table>
<thead>
<tr>
<th>Major subject</th>
<th>Future profession</th>
<th>Mathematics studies (ECTS*)</th>
<th>Pedagogical studies (ECTS)</th>
<th>Other subjects (ECTS)</th>
<th>Total (ECTS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics major</td>
<td>Mathematician</td>
<td>180</td>
<td>0</td>
<td>120</td>
<td>300</td>
</tr>
<tr>
<td></td>
<td>Subject teacher</td>
<td>120</td>
<td>60</td>
<td>120</td>
<td>300</td>
</tr>
<tr>
<td></td>
<td>Hybrid teacher</td>
<td>120</td>
<td>120</td>
<td>60</td>
<td>300</td>
</tr>
<tr>
<td>Mathematics minor</td>
<td>Subject teacher</td>
<td>60</td>
<td>60</td>
<td>180</td>
<td>300</td>
</tr>
<tr>
<td></td>
<td>Hybrid teacher</td>
<td>60</td>
<td>120</td>
<td>120</td>
<td>300</td>
</tr>
</tbody>
</table>

*ECTS refers to the European Credit Transfer System. One ECTS point is the equivalent of 25 hours of study. The recommendation is to complete 60 ECTS of studies per year. All MS degrees can be completed in 5 years.
For their future professions, mathematics majors can choose between mathematician, subject teacher, and hybrid teacher, while minors can choose between subject teacher and hybrid teacher. Subject teachers and hybrid teachers both take the same Subject Teacher’s Pedagogical Studies (60 ECTS), but, in addition, hybrid teachers also take the Multidisciplinary Study Programme in Basic Education (60 ECTS). The Multidisciplinary Study Programme in Basic Education studies is required for working at primary schools, while and Subject Teacher’s Pedagogical Studies are required for working as a subject teacher. The hybrid teachers have a dual qualification for working as a class teacher or a subject teacher at any school level in Finland, whereas the subject teachers can work as subject teachers at any school level except primary schools. All degrees are MS degrees and they require studies for a total of 300 ECTS points.

**Mathematics studies.** Mathematics majors and minors both take the same basic and intermediate mathematical studies (60 ECTS); in addition, mathematics majors take also advanced-level mathematics studies (60 ECTS). Most basic and intermediate mathematical studies are traditional mathematics courses focusing on different mathematical topics, such as calculus, algebra, differential equations, and geometry. The courses are usually suitable for both future teachers and mathematicians, and hence the course work is for the most part the same for the future teachers and the future mathematicians. Some mathematics courses at intermediate and advanced levels are designed especially for future teachers. These courses focus, for example, on the history of mathematics, visualizing mathematics, school mathematics, mathematical thinking, and technology in teaching. Since mathematics majors have extensive mathematical studies, they also have more freedom to choose their mathematical studies and therefore, in contrast to mathematics minors, mathematics majors usually take more mathematical studies that are designed in particular for teachers. It is notable that mathematics majors and minors have the same competence to teach mathematics at the secondary school level even though
mathematics majors take twice the number of mathematics courses in comparison to mathematics minors (see Table 9).

*Pedagogical studies.* Subject Teacher’s Pedagogical Studies (60 ECTS) are required for working as a qualified teacher in Finland. Subject Teacher’s Pedagogical Studies (60 ECTS) focus on teaching and learning in general (30 ECTS), the didactics of mathematics (10 ECTS), and teaching practice (20 ECTS). The Multidisciplinary Study Programme in Basic Education (60 ECTS) are included only for hybrid teachers, and they focus on interaction (15 ECTS), the learning community (20 ECTS), participation and society (20 ECTS), and expression and culture (5 ECTS). The Multidisciplinary Study Programme in Basic Education emphasizes the characteristics of many school subjects, and therefore the studies prepare teachers to gain an understanding of teaching and learning at primary level.

Teaching practice is mainly undertaken at the university teacher training school. In the course of their teaching practice, the student teachers plan their own teaching sequences or lessons under the guidance of a subject teacher. The student teachers’ lessons are evaluated and feedback is also provided. The training school teachers’ task is to guide student teachers in addition to performing their own ordinary teaching work. The amount of student teaching is approximately 50 lessons.

*Other subjects.* All degrees also include the freedom to choose another minor subject to study (see Table 9). Students can choose any other subject that they wish from approximately one hundred different choices at the University of Eastern Finland. The typical choices for future teachers are nevertheless physics or chemistry, or both, because these subjects are usually required for teaching positions at school level. In light of all of these factors, it can be said that future teachers will have the competence to teach mathematics and other subjects at lower or upper secondary schools and vocational schools.
3.5 DATA COLLECTION AND SAMPLE

All data collection was conducted under the auspices of the University of Eastern Finland. Teacher educators and graduated teachers participated in the survey study, and student teachers took part in the essay study.

3.5.1 Survey

The survey data was collected in 2012-2013. The first survey targeted all of the mathematics teachers who had graduated from the University of Eastern Finland in 2002-2012. Hybrid teacher programs were not offered in 2002-2012, and hence all of the graduated teachers were subject teachers. Our sample (N=101) included 54% of all of the teachers who had graduated in the course of 2002-2012. 72% (73) of the sample were mathematics majors, while 28% (28) were mathematics minors. All of the respondents, with one exception, had previous experience of working as mathematics school teachers or they were subject teachers in mathematics at the time of the data-collection. The respondents’ gender and major subject were similar in the sample to those of all of the potential respondents (Figure 7).

Figure 7. The respondents’ gender and major subject were similar in the sample to those of all of the potential respondents and included 54% of all graduates in the period 2002-2012.
The second survey targeted teacher educators who taught courses in mathematics or pedagogical studies, or who were guiding teaching practice within the mathematics teacher education program at the University of Eastern Finland in 2012. Our sample (N=19) included 79% of all the teacher educators who were contracted to work in the mathematics teacher education program in 2012. 74% (14) of the sample taught mathematics (Math-Educators), while 26% (5) worked in pedagogical studies or teaching practice (PT-Educators). To conceal the respondents' identities, the teacher educators involved in both pedagogical studies and teaching practice were placed in the same category. One PT-Educator responded to open questions but not to the 72 items, and so our sample (N=18) for the 72 items included 75% of all of the teacher educators, where the rates were 78% (14) for the Math-Educators and 22% (4) for the PT-Educators. The respondents’ gender and teaching areas were similar in the sample to those of all of the potential respondents (Figure 8).

Figure 8. The respondents’ gender and teaching areas were similar in the sample to those of all of the potential respondents and included 79% of all those teaching courses in mathematics or pedagogics, or were guiding teaching practice at the University of Eastern Finland.

### 3.5.2 Essays

The essay data was collected in 2015 during a pilot course called *Analysis skills for teaching mathematics*. The course was an advanced level mathematics course and thus aimed especially at
mathematics majors (i.e. future subject teachers and hybrid teachers).

The student teachers wrote essays under the rubric of the kind of knowledge needed for teaching mathematics. They were then informed that one of the aims of the pilot course was that they should learn to reflect on their own ideas related to the knowledge needed for teaching mathematics. The essays were returned to the student teachers and they were asked to share their thoughts and discuss the essay topic in small groups. In addition, they were also requested to make notes about these conversations, and the course instructor encouraged them to “steal” each others’ ideas, but only if the stolen ideas matched their own thinking. These small group sessions lasted two hours and were repeated a total of three times during the pilot course, each time with mixed small groups, which meant that each student teacher had the opportunity to engage in a discussion with everyone else at least once. At the end of course, the student teachers were asked to reflect on their ideas, but this time individually, by writing an essay about the kind of knowledge needed for teaching. Finally, these essays produced by the student teachers wrote analyzed.

It is entirely possible that the student teachers learned something new about teacher knowledge during the pilot course. However, in each situation the course instructor emphasized that the student teachers must reflect on their own thinking and assess all new ideas. In addition, the contents of the pilot course were related to the topic of graph theory and analysis of the teaching of graph theory, and the teacher did not refer to teacher knowledge itself throughout the course.

Eighteen student teachers participated in a course devoted to the Analysis skills required for teaching mathematics, and all of them also participated in the ongoing study. Eight of the participants were future subject teachers, while ten of them were future hybrid teachers. At the stage at which they participated in the course, they had completed almost all of the compulsory
mathematical studies and over half of their pedagogical studies and teaching practice (Figure 9).

![Completed studies chart](image)

*Figure 9. The background details of the future subject teachers and hybrid teachers were similar.*

In addition, some of the student teachers in both groups had already acquired some experience of working in schools. On average, the participants were in their final year of study at the time of the collection of the data. Both of the student teacher groups were at a relatively similar stage in their studies, and hence we paid no attention to any of the small differences between the two groups.

### 3.6 ANALYSIS

Similar content analysis methods were used for analyzing the essays and open-ended questions in the survey. In addition, the essays were analyzed with the aid of network analysis. Quantitative methods were used to analyze the seventy-two statements in the survey.

#### 3.6.1 Survey

The seventy-two items were analyzed as follows. Four mean values were calculated for each statement. One mean was calculated on the basis of the graduated teachers survey, summing up the perceptions held by all of the graduated teachers.
regarding the way in which they had learned each of the topics mentioned in the survey. It is reasonable to assume that the aims of the course are different in mathematics and pedagogical studies, and hence three means were calculated on the basis of the teacher educators survey. The first summarized the perceptions of the Math-Educators, the second the perceptions of the PT-Educators, and the third the perceptions of all of the teacher educators concerning the ways in which their students had learned each of the topics. The teacher educators’ views are presented separately although there were only a handful of educators involved in the pedagogical studies and teaching practice.

It was presumed that the mathematics majors and minors would view their learning differently because their course work was different. The number of mathematical courses taken by mathematics majors is, for example, double that taken by mathematics minors. The Mann-Whitney U-Test assumes that both groups are similar (Sheskin, 2003). According to Sheskin (2003) “If the result of the Mann-Whitney U-test is significant, it indicates there is a significant difference between two sample medians, and as a result of the latter the researcher can conclude there is a high likelihood that the samples represent populations with different median values” (p. 423). Thus, the Mann-Whitney U-Test was appropriate for exploring differences between the mathematics majors’ and minors’ perceptions, since the study groups were not dependent on each other.

Cronbach Alpha test can be used for testing the consistence of an instrument. According to Cronbach (1951) higher Cronbach Alpha scores indicate stronger correlation between tested items. Survey items were designed in the way that each statement was related to one of the MKT domain and therefore the consistency of statements were tested within these categories. Since the number of responses was fivefold in graduated teachers’ survey, their responses were selected to test the consistency. The Cronbach’s Alpha scores were .931 for the items in subject matter knowledge (CCK .938, SCK .736, HCK .781), and .950 for the items
in pedagogical content knowledge (KCT .885, KCS .921, KCC .668). The results show that items correlated highly within the six domains of MKT, which indicates that the variation between the items in these categories were relatively small (Tavakol & Dennick, 2011). In addition, the Cronbach’s Alpha score .931 in subject matter area produced a .13 error variance (random error) and an Alpha score of .95 in pedagogical content knowledge produced a .10 error variance (see Tavakol & Dennick, 2011).

The teacher educators survey and graduated teachers survey included the same 72 items. Both of the respondent groups expressed their opinions about the ways in which the graduated teachers had learned a number of different issues. It was not, however, self-evident that these perceptions would be similar, and hence it was decided to check the correlation between their views. This was calculated solely on the basis of the mean values. In another words, the correlation between two mean values was checked for all 72 items. A proper method for calculating correlations depends on the normal distribution of data. Normally, there are many ways for testing distribution, e.g. using tests such as those referred to as Shapiro-Wilk, Kolmogorov-Smirnov, Lilliefors, and Anderson-Darling (Razali & Wah, 2011). The study published by Razali and Wah (2011) claims that the most powerful test for all types of distribution and sample sizes is the Shapiro-Wilk test. In consequence, the normal distribution of data was checked using the Shapiro-Wilk test and, in addition, also using the Komogorov-Smirnov test, even though it is a somewhat less powerful test (Razali & Wah, 2011). Since both tests indicated that the data was normally distributed, use was made of Pearson’s correlation (Hauke & Kossowski, 2011).

To analyze the open-ended questions two recognized content analysis approaches were used (Hsieh & Shannon, 2005). After several readings, we noted that respondents’ personal experience of, or attitudes toward, the contents of their mathematics teacher education emerged from the data with considerable clarity. To investigate these personal experiences of, or attitudes toward, the
Categorizing respondents into positive, negative, analytical, or neutral based on their attitudes and issues mentioned, a significant number of respondents mentioned issues that required development, which were classified as “in need of development.”

To further investigate the nature of these issues, direct content analysis was employed. This approach begins with theory, and theory-based categories are utilized as a guide in the analysis (Hsieh & Shannon, 2005). In this case, the MKT domains served as the basis for classifying the issues into six distinct categories. The methodology used for classifying topics into MKT domains was comparable to that utilized in the Markworth, Goodwin, and Glisson (2009) study.

Markworth et al. (2009) employed MKT to analyze conversational topics and interview responses, enabling them to gauge the nature of the topics learned by student teachers during a teaching practice course. They concluded that using MKT in their analysis allowed them to capture comprehensive insights into the subject matter knowledge and pedagogical content knowledge acquired during the teaching practice course.

In our study, the use of MKT in direct content analysis was a prudent choice, given that MKT had already been applied as the basis of the statements, and preliminary analysis confirmed that almost all of the topics could be categorized into the six MKT domains.
3.6.2 Essays

Eighteen essays under the rubric of The knowledge required for teaching mathematics contained an average of 428 words each (maximum 833; minimum 249; standard deviation 160), and they were analyzed as follows. First, Conventional content analysis was used in order to place the knowledge issues alluded to into data-based categories (Hsieh & Shannon, 2005). For example, respondents mentioned that teachers needed to “know mathematical theories”, which was classified as a single data-based category. Next, Direct content analysis was used to classifying the data-based categories into the six domains of MKT (Hsieh & Shannon, 2005). At the same time, we tracked the ways in which topics were related to each other (see Table 10). Our analysis yielded a total of 136 knowledge issues and 364 relations.

Table 10. The contents of essays were transformed into networks.

<table>
<thead>
<tr>
<th>Sample section of essay</th>
<th>Data-based classification (Conventional Content Analysis)</th>
<th>Theory-based classification (Direct Content analysis)</th>
<th>Network</th>
</tr>
</thead>
<tbody>
<tr>
<td>A teacher should make mathematics meaningful. A teacher should connect the contents he/she is trying to teach to everyday life and also provide more information about the history of mathematics.</td>
<td>Explains why mathematics is needed</td>
<td>Specialized Content Knowledge</td>
<td>Create connections to everyday life</td>
</tr>
<tr>
<td></td>
<td>Create connections to everyday life</td>
<td>Specialized Content Knowledge</td>
<td>Explains why mathematics is needed</td>
</tr>
<tr>
<td></td>
<td>Create connections to the history of mathematics</td>
<td>Specialized Content Knowledge</td>
<td>Create connections to history of mathematics</td>
</tr>
<tr>
<td>A teacher needs knowledge about the structure of</td>
<td>Know the structure of mathematics</td>
<td>Horizon Content Knowledge</td>
<td></td>
</tr>
</tbody>
</table>
mathematics, because he/she needs to know what prior knowledge the new learning requires.

Know the prior knowledge in the learning of a new topic

Horizon Content Knowledge

Self-evidently, subject knowledge plays a major role. You cannot teach mathematics if you do not know mathematics. At school level, there are so many different topics, and a teacher needs to know them and to use that knowledge.

Know mathematical contents to be taught in school

Common Content Knowledge

The data was then converted into Gephi format. Gephi is software which can be used in exploring, manipulating, and visualizing large networks (Bastian et al., 2009). A network is a mathematical construction made up of vertices, nodes, or points that are connected with edges, arcs, arrows, or lines. In the present study, all of the edges were directed, and therefore we prefer to use the terms nodes and arrows. Nodes are knowledge topics mentioned by the student teachers in their essays, while arrows mark the ways in which the knowledge topics were connected to each other (see the final column in Table 10). In the network analysis, out-degree refers to the number of outward-directed arrows from each node, while in-degree refers to the number of inward-directed arrows attached to each node. Once the data had been imported into the Gephi software, one large network resulted, consisting of 136 nodes and 364 arrows between the nodes. Because the large network was constructed on the basis of eighteen student teachers’ ideas, it may be said to efficiently represent the teacher knowledge present in the minds of the future teachers.

For examining the ways in which the knowledge topics were connected in the network, we investigated how strongly the
different parts of the network were connected. In the context of
the network analysis, the strongly connected parts of the network
are referred to as communities. When using the Gephi software, a
modularity can be used to detect communities within the network
(Blondel, Guillaume, Lambiotte, & Lefebvre, 2008). A resolution in
the modularity count is a value used in optimizing the number of
communities, which means that a lower resolution can be used in
order to achieve more communities, while, conversely, a higher
resolution can be used to achieve fewer communities. A standard
resolution of 1.0 was used in our study (Lambiotte, Delvenne, &
Barahona, 2009). In addition, the large network represents the
teacher knowledge based on the eighteen student teachers’ ideas,
while the network communities represent the various sections of
the teacher knowledge. Hence, in the present study the term
section is used in relation to teacher knowledge rather than to the
community or modularity class.

To investigate the ways in which the various sections of teacher
knowledge were interconnected, we used the Gephi Atlas2
algorithm. The Atlas2 algorithm separates the weakly connected
parts, while it keeps the strongly connected parts together
(Jacomy, Venturini, Heymann, & Bastian, 2014). Because the
modularity count identifies communities and Atlas2 manipulates
the layout to show how the various communities are related, it is
reasonable to use both of them to explore the ways in which the
communities are arranged in the network. In order to investigate
in more detail how the various topics were interconnected, we
used a partition parameter and a grouping tool in Gephi. A
partition parameter can be used to identify the properties of
nodes or arrows in Gephi that are similar (see Khokhar, 2015).
Because all of the nodes had been classified into the six domains
of MKT, we were able to sort all of the nodes into six groups based
on their MKT classification with the of the partition parameter.
The grouping tool was then used to investigate how the six node
groups, in this case the MKT domains, were interrelated. By using
the grouping tool, we were also able to investigate the hierarchy
of these six node groups.
4 Main results

The combined survey findings revealed the need for some developments in the content of the mathematics teacher education program that was under study. These findings were reported in Articles I and II. To produce a consistent picture of these received development needs, the results of Articles I and II will be discussed together in section 4.1. Student teachers’ perceptions of the knowledge required in teaching mathematics were studied and analyzed with the aid of network analysis. The findings, reported in Article III, will be discussed in section 4.2.

4.1 TEACHERS’ AND THEIR EDUCATORS’ PERCEPTIONS OF DEVELOPMENT NEEDS IN THE CONTENTS OF MATHEMATICS TEACHER EDUCATION

Both the teacher educators’ and the graduated teachers’ surveys contained the same seventy-two statements. These statements were concerned with the knowledge required to for teach mathematics, and all were related to the six MKT domains. The teacher educators evaluated the graduated teachers’ learning based on their course contents, while the graduated teachers evaluated their own learning based on the totality of their teacher education. The mean values for their perceptions of graduated teachers’ learning are shown in Figure 10.
In Figure 10, the red circles are higher than the blue ones, which means that the teacher educators in pedagogical studies and teaching practice (PT-Educators) rated the graduated teachers’ learning somewhat more positively than the teacher educators in mathematical studies (Math-Educators). The teacher educators evaluated their students’ learning based on their course contents, and the higher ratings may indicate an unrealistic conception of their students’ learning or the diverse aims of course contents. However, all of the teacher educators and graduated teachers generally regarded the graduated teachers’ learning similarly (see the grey circles and bars in Figure 10). Figure 11 suggests that all of the teacher educators and all of the graduated teachers seem in all seventy-two items to have more stabilized perceptions of the graduated teachers’ learning. Pearson’s correlation supports this conclusion, since for all 72 of the items, there was significant positive linear correlation ($r=.502$, $p<.000007$). Thus, it can be said that, on average, the perceptions of the teacher educators and
graduated teachers about the graduated teachers’ learning were similar.

Overall, the graduated teachers’ mean for items in subject matter knowledge was 3.24, while for items in pedagogical content knowledge it was 2.67. The teacher educators’ means were, in overall terms, 2.98 for items falling under subject matter knowledge and 2.70 for items in pedagogical content knowledge. Both of the study groups shared the opinion that graduated teachers have learned topics slightly better in the area of subject matter knowledge than that of pedagogical content knowledge. By examining subject matter knowledge and pedagogical content knowledge in greater detail, the six MTK domains reveal scores, for both study groups, that for Common Content Knowledge are highest and for Specialized Content Knowledge lowest in terms of subject matter knowledge. Furthermore, the graduated teachers’ means are highest for items in Knowledge of Content and Teaching and lowest for items in Knowledge of Content and
Students, in terms of pedagogical content knowledge. Hence, in what follows, we take a more detailed look at the six MKT domains and, through them, explore the nature of the development requirements that exist in the contents of the mathematics teacher education program under study.

*Common Content Knowledge.* The graduated teachers considered that the present contents should be developed because mathematics are not same at university as the topics that need to be dealt with in schools. They felt that some mathematical topics either remained underemphasized or were not treated at all during their education (Article I). They also suggested that they had learned modest mathematical topics such as geometry and statistics (Article II), while, in addition, the graduated teachers were of the opinion that they needed a more developed knowledge of geometry, statistics, and financial mathematics (Article I). These teachers claimed that they required greater knowledge of these topics because after graduation they had needed to study these topics in greater depth in order to be ready to teach (Article I). The findings reported in Articles I and II also suggest that future teachers required more mathematical knowledge related to geometry and statistics. These mathematical topics are compulsory at Finnish school level, but they nevertheless were topics that the graduated teachers felt that they had been too little exposed to. Similarly, some of the teacher educators felt that the future teachers required more knowledge of pure mathematics, although they did not name the specific mathematical topics that future teachers actually lacked (Article I). They also suggested that a knowledge of mathematical topics is fundamental for teaching and that knowing too much could never be harmful. However, the graduated teachers did not entirely agree with this perception. They were disappointed that university mathematics appeared to overemphasize topics for which they had never had any use in their school teaching (Article I). In addition, they expressed the criticism that the present contents of what they had learnt were not well linked with school mathematics. By this they meant that even though the contents of what taught in university and school mathematics
were similar, they were unable to see the precise connection between university mathematics and the subject contents taught at school level (Article I). They argued that the presentations made at university were sometimes too symbolic, theoretical, or complex compared with the mathematics taught at schools and hence they were unable to see the necessary connections.

Specialized Content Knowledge. Both the teacher educators and the graduated teachers were of the opinion that the graduated teachers had learned generally somewhat modest topics in Specialized Content Knowledge, especially in comparison to the other components of subject matter knowledge (Article II). To be more precise, the graduated teachers felt that they had only a modest knowledge of, for example, formulating and marking exams, the history and philosophy of mathematics, or knowing how mathematics has influenced everyday life (Article II). The results indicate that Specialized Content Knowledge could be less emphasized than Common Content Knowledge. Article I indicates the same, since the graduated teachers proposed that the present course contents for mathematical studies should differ for future teachers and future mathematicians. They claimed that the present mathematical contents focus too extensively on presenting and proving mathematical theorems, which are appropriate contents for future mathematicians but whose ramifications do not take into account the special requirements of the teacher’s profession. Secondly, the graduated teachers felt that the current amount of school mathematics should be increased or that school mathematics should in future be taken better into account. In addition, the teacher educators considered that the present mathematics courses were not working well for future teachers, and therefore the contents should be revised in light of the teaching professions (Article I). In addition, the teacher educators also suggested that mathematics courses should be at least partly designed separately for future teachers and future mathematicians. The findings reported in Articles I and II suggest that the teacher educators and the graduated teachers agree that the present mathematical contents should be revised with an eye on the
mathematics teaching profession. In addition, school mathematics and Specialized Content Knowledge should also in future be more fully taken into account.

Knowledge of Content and Teaching. The graduated teachers felt that they had made good use of their knowledge in planning single lessons but that they were less successful in planning complete courses (Article II). Article I indicates the same: future teachers require a more developed knowledge relevant to planning complete courses. Furthermore, the graduated teachers proposed that future teachers would need more general knowledge about the pedagogy of mathematics and the ways in which students could be better motivated to learn mathematics (Article I). In addition, they suggested that they need a wider range of skills to adapt their teaching to make it relevant, for example, to both “good and weak students”. The findings reported in Articles I and II suggest that future teachers require more extensive knowledge relevant to planning whole courses, knowledge about motivating students, and generally knowledge about different teaching methods, which was sometimes referred to generally as didactic mathematics.

Knowledge of Content and Students. The graduated teachers considered that in general they had learned a modest amount of Knowledge of Content and Students in comparison to other components in the field of pedagogical content knowledge (Article II). Furthermore, they felt they had acquired a modest knowledge of students’ challenges, such as learning difficulties and misconceptions (Article II). This topic is also repeated in Article I. The graduated teachers felt that they needed more knowledge related to learning difficulties, evaluating student knowing, and handling students at different levels (Article I). They suggested that in the course of their own teaching they had witnessed students struggling with mathematics and those students may even have had learning difficulties, but that they, as teachers, had insufficient skills to recognize and support those students. The findings reported in Articles I and II suggest that future teachers require a greater degree of knowledge concerning
students’ misconceptions, learning difficulties, and other challenges encountered in learning mathematics, identifying students’ mathematical abilities, and understanding how students learn mathematics, i.e., they needed more knowledge related to the different ways to learn mathematics.

Differences between the perceptions of mathematics majors and minors. Mathematics majors and minors had relatively different curricula in the mathematics teachers’ education program under study, and it was for this reason that the differences between their views about their own learning were investigated (Article II). The mathematics majors rated their own learning more highly than the minors in thirteen statements. With the exception of one, all the statements focused on subject matter knowledge and most of the differences existed in the various domains of Common Content Knowledge. The number of mathematical courses taken by the majors was double that taken by the minors in the mathematics teacher education program under study, which seems to be a reasonable explanation for the differences. Interestingly, the majors felt that they had a better knowledge of geometry, number sequences, and the history of mathematics, which were optional courses in the mathematics teacher education program, which were taken by most of the mathematics majors. Thus, it may be claimed that the different curricula offered to the mathematics majors and minors is a reasonable explanation for the differences.

The demand for development. The respondents’ personal experience of or attitudes toward the contents of mathematics teacher education was also investigated (Article I). The results show that only a minority of graduated teachers were satisfied with the present course contents. Fifty-nine percent of the graduated teachers were willing to develop the present mathematics courses, while 67 percent were willing to develop the pedagogical courses and teaching practice included in the mathematics teacher education program under study. The teacher educators’ perceptions reflect a similar demand for development to be undertaken. Only 26 percent of the teacher educators were fully
satisfied with the present contents of their courses. The combined attitudes of the teacher educators and graduated teachers’ reflect the observation that there is clearly a demand for the contents of the mathematics teacher education program to undergo development. Their suggestions regarding the development of the whole mathematics teacher education program included suggestions that were similar to their ideas about how the mathematics or pedagogical courses should be developed. They suggested, for example, that the present contents of the mathematics should be revised so that it would become more useful for future teachers; that the mathematics courses should be designed separately for teachers and mathematicians; that subject knowledge and pedagogical knowledge should be blended; that the sequencing of contents could be usefully re-organizing so that the links between university and school mathematics would become more explicit, by demonstrating the links between theory and practice, and, finally, that the amount of teaching practice should be increased (Article I).

4.1.1 Summary of the development needs of mathematics teacher education

The findings reported in Articles I and II are consistent and indicate a similar range of development needs for the mathematics teacher education program. The qualitative findings reported in Article I generally provided a great deal of information about the kinds of development needed. However, the quantitative results reported in Article II also confirmed or supplemented these findings. By using the same theoretical framework we were able to explore both the teacher educators’ and the graduated teachers’ perceptions from similar angles. The main development requirements in the contents of the mathematics teacher education program under study are related to the four MKT domains and also to the problems involved in linking theory to practice (Table 11).
As Table 11 demonstrates, the issues raised are quite common, and it is possible that this kind of challenge could even be universal in the context of mathematics teacher education. The generalizations of our findings require more research. After the survey study, we focused on studying relationships in teacher knowledge by investigating how future teachers view the knowledge required in teaching mathematics.

### Table 11. Summary of the main findings reported in Articles I and II.

<table>
<thead>
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<th>The summary</th>
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<tr>
<td><strong>Common Content Knowledge</strong></td>
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<tr>
<td>• University mathematics should cover all topics taught at school level.</td>
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<tr>
<td>• University mathematics should have clear connections with school mathematics.</td>
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<tr>
<td><strong>Specialized Content Knowledge</strong></td>
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<tr>
<td>• Some mathematical contents in teacher education should be designed only for future teachers.</td>
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<tr>
<td>• These mathematical contents should take account of school mathematics</td>
</tr>
<tr>
<td>• The mathematical contents should be designed so that they take account of the special characteristics of the teaching profession.</td>
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<tr>
<td><strong>Knowledge of Content and Students</strong></td>
</tr>
<tr>
<td>• The pedagogical contents included in teacher education should take into account that teachers require knowledge that will enable them to recognize their students’ abilities, e.g. by identifying “good and weak students”</td>
</tr>
<tr>
<td>• Teacher education should prepare teachers so that they can understand how students learn mathematics, i.e. by determining the different ways in which mathematics can be learned.</td>
</tr>
<tr>
<td>• The pedagogical contents should prepare teachers to become aware of students’ misconceptions, learning difficulties, and any other challenges that their students might encounter while learning mathematics.</td>
</tr>
<tr>
<td><strong>Knowledge of Content and Teaching</strong></td>
</tr>
<tr>
<td>• The pedagogical contents of teacher education should prepare teachers so that they can teach a wide variety of students at the same time, i.e. both “good and weak students”.</td>
</tr>
<tr>
<td>• Since student groups are always heterogeneous, teacher education should prepare teachers to differentiate their teaching.</td>
</tr>
<tr>
<td>• The pedagogical contents should take teachers’ knowledge of how to motivate students in learning mathematics into account.</td>
</tr>
<tr>
<td>• In addition to teachers needing knowledge concerned with planning single lessons, teacher education should also prepare them to be able to undertake the planning of complete courses.</td>
</tr>
<tr>
<td><strong>Linking theory to practise</strong></td>
</tr>
<tr>
<td>• The mathematical and pedagogical contents in teacher education may be experienced as both too theoretical and also too weakly linked to practise.</td>
</tr>
<tr>
<td>• The mathematical and pedagogical contents included in teacher education may be experienced as poorly integrated.</td>
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</table>
4.2 STUDENT TEACHERS’ PERCEPTIONS OF THE KNOWLEDGE REQUIRED FOR TEACHING MATHEMATICS

In the second phase of the research, an investigation was conducted into student teachers’ perceptions of the knowledge required for teaching mathematics, and especially the linkages between the various areas of knowledge (Article III). Eighteen essays titled *Knowledge needed for teaching mathematics* were transformed into a single large network. Since the single large network is made up of the perspectives of eighteen student teachers, it refers in a thorough manner to the teacher knowledge of the student teachers involved in the research. To investigate which parts of the network were strongly connected to each other, modularity counting facilitated by the Gephi software was utilized. Eighteen different sections of the teacher knowledge were identified as relevant (with a modularity of 0.601). Eight sections were smaller, and contained only between one and four knowledge topics and therefore they were presented as a single composite topic.

Overall, the large network contained teacher knowledge in the form of eleven different sections. These are presented and discussed in detail in the results section of Article III (pp. 13-25). According to the results, the student teachers’ perceptions of the knowledge required for teaching mathematics can be summarized in terms of the eleven sections of teacher knowledge:

1. Knowing mathematics
2. Linking mathematics
3. Understanding the structural aspects of learning mathematics
4. Knowing the common challenges in mathematics
5. Knowing different teaching methods
6. Choosing appropriate teaching methods
7. Promoting students’ different approaches to learning
8. Separating teaching
9. Following the guidelines of the national curriculum
10. Using technology
11. Assessing and improving teaching

In light of these eleven sections, it is pertinent to examine two examples of them in closer detail. The student teachers suggested that teachers required a knowledge of mathematical terms, rules, concepts, symbols, logic, and theories (see the left side of Figure 12, below). Using this knowledge, a teacher will be able to discuss mathematics, advise students during lessons, justify and prove mathematical theorems, apply mathematics, evaluate the ways in which exercises work in practice, and generally inspire students (see the right side of Figure 12). It makes sense that these issues are connected as student teachers mentioned but it should be noticed that jumps occur from one MKT domain to another. The topics on the left consist of background knowledge and are related to knowing mathematics, which can be classified as Common Content Knowledge. In contrast, the topics listed on the right are related to several MKT domains. In consequence, this sub-network was categorized as Knowing mathematics, forming the first section of teacher knowledge.

![Diagram](image.png)

*Figure 12. Mathematical theories are required in proving, discussing, and applying mathematics (Article III).*
The student teachers claimed that teachers should create connections between mathematics and other school subjects, everyday life, and the history of mathematics (see left side of Figure 13). With the aid of this kind of knowledge, teachers can explain why and where mathematics is required and they can also promote students’ understanding of mathematics (see right side of Figure 13). For example, to promote students’ understanding, teachers should explain mathematics in a variety of ways, and to explain mathematics in a variety of ways, teachers should know the mathematical topics that they teach. This, in turn, requires a firm knowledge of the contents of school mathematics and also an ability to study mathematics. Because many of these issues are linked with the creation of connections and with the production of storylines around mathematics, this second section of teacher knowledge has been labeled *Linking mathematics*.

![Figure 13. The connections between mathematics and everyday life should be established in a way that will explain why mathematics is needed (Article III).](image)

To examine how the eleven sections of teacher knowledge are interconnected, we used the Atlas2 algorithm to manipulate the layout of the large network. The Atlas2 algorithm separates weakly connected parts of the network while keeping the more strongly connected part together. The results yielded by the network were the sections based on modularity counting, which are distinguished by the different colors in the Figure 14.
Figure 14. Eleven different components of teacher knowledge were identified (Article III).

Figure 14 shows that Atlas2 has separated the weakly connected parts while keeping the strongly connected parts of the network together (sections of teacher knowledge). Hence, the sections that have more probable connections with each other are neighbors in this layout diagram representing the network. Figure 14 shows, for example, that Knowing mathematics is closely related to Linking mathematics. This indicates that teachers should know mathematics in order to create connections between mathematics and other topics. Similarly, the section Knowing the common challenges in mathematics is related to the two sections Choosing appropriate teaching methods and Separating teaching. This suggests, in turn, that knowing the common challenges that may be hidden in mathematics impact on how teaching methods are selected, and also, separating teaching may well be a potential solution that takes account of those students who struggle with learning mathematics. Furthermore, the section Using technology is connected both to the section Knowing different teaching methods and also to Promoting students’ different ways of learning. This
indicates that a teacher requires specific knowledge in order to be able to integrate technology with teaching methods. In addition, technology can also be used to promote students’ different ways of learning.

Since Atlas2 separates the weakly connected parts of the network, the center of the network consists of parts that are strongly connected to each other (Figure 14). The center of network is made up of two sections, Promoting students’ different ways of learning and Choosing appropriate teaching methods. The result suggests that almost all of the sections of teacher knowledge are connected in some way to these two sections. This indicates that almost all of the sections of teacher knowledge are related to the answer to such questions as “How can an appropriate teaching method be selected?” or “How can different ways of learning be promoted?”

One section, identified as Following the guidelines of national curriculum, nevertheless remains unconnected to any other sections of teacher knowledge. This indicates that the student teachers considered that the process of following the guidelines of the national curriculum is an independent part of teacher knowledge that is unconnected with the other sections of teacher knowledge.

Up to this point, the results indicate that the eleven sections of teacher knowledge do not match the various domains of MKT exactly. Rather, it can be clearly seen that, in the minds of the student teachers, numerous jumps occur from one MKT domain to another in each section of teacher knowledge. Furthermore, we have also observed that the sections of teacher knowledge are connected to each other in a variety of ways, while almost all of the sections are connected to the center of a large network. All of these results indicate that student teachers participating in the present study do not consider the knowledge related to the six domains of MKT as independent, but the six MKT domains are nevertheless interconnected in various ways in the minds of the student teachers.
In order to examine in greater detail how the knowledge topics are interrelated, we made use of a Gephi partition parameter and a Gephi grouping tool. First, the partition parameter was used to sort all the knowledge topics into six groups based on their MKT classification. Then the grouping tool was used to examine the structure of network. This procedure yielded a network that expresses how the knowledge topics, classified in terms of the six MKT domains, are indeed interrelated (Figure 15). The node size illustrates the indegree value of the specific node in Figure 15.

Figure 15 reveals that Common Content Knowledge has an outdegree of thirty but an indegree of zero. This means that the student teachers always discussed the topics related to Common Content Knowledge in the understanding that this knowledge...
was required in order for something to be undertaken. The result indicates that in the minds of the respondents Common Content Knowledge is more like the prerequisite background knowledge for teaching. However, Knowledge of Content and Curriculum similarly has an outdegree of thirty-seven but an indegree of only three. This indicates that Knowledge of Content and Curriculum may also be regarded as background knowledge and thus required so that something can be undertaken.

The indegree and outdegree values are balanced with regard to Horizon Content Knowledge and Specialized Content Knowledge (Figure 15). The same number of arrows are directed inside and outside of both of the domains. The results suggest that some knowledge is needed in order to understand the Horizon Content Knowledge and Specialized Content Knowledge, but these knowledge types are also required for undertaking something. This is pertinent because the student teachers claimed that teachers need background knowledge of mathematical theories to be able to draw up connections, for example, between mathematics and everyday life. Based on these views, during this process teachers convert their subject knowledge into a form that will help students understand why and where mathematics is needed. Teachers transform subject knowledge from its form as it exists in Common Content Knowledge into a form of Specialized Content Knowledge. This process requires background knowledge about the subject but also knowledge concerning the special characteristics of mathematics. Similarly, the student teachers suggested that a teacher needs to master the various ways of using technology and other teaching aids, which can therefore be classified as familiarity with Knowledge of Content and Curriculum. Furthermore, the student teachers thought that teaching aids and technology may be used to illustrate mathematics. A similar transformation occurs in this phase. The process begins with mastering teaching aids and continues to the selection of suitable ways to use the aids, but when teachers attempt to illustrate mathematics knowledge itself, knowledge of the particular characteristics of mathematics becomes a prerequisite.
Figure 15 indicates that, based on student teachers’ perceptions, Knowledge of Content and Students and Knowledge of Content and Teaching are the final phase in the totality of teacher knowledge. Knowledge related to the four MKT domains is all, more or less, connected with KCT and KCS. In addition, these two domains are also strongly interconnected. This is pertinent since the student teachers were of the opinion that learning theories, for example, or empirical observations concerning their students can be effective starting-points in the selection of appropriate teaching methods. Teachers may recognize the challenges inherent in student learning or they may note that students struggle while learning mathematics (in this case, Specialized Content Knowledge will also be involved), which has an effect on the kind of decisions that the teacher will make in those situations. According to the results, decisions that include the selection of appropriate teacher methods and of promoting students’ different ways of learning are related to almost all of the sections, and hence the two questions, “How should appropriate teaching methods be selected?” and “How can different learning be supported?” also activate all of the domains of MKT.

These results show that, in the minds of the student teachers investigated here, there exists an MKT hierarchy. Common Content Knowledge and Knowledge of Content and Curriculum are, above all, Foundation knowledge (connections mostly external), while the domains of Horizon Content Knowledge and Specialized Content Knowledge are primarily Transformation knowledge (in- and out-connections in balance) and the domains of Knowledge of Content and Students and Knowledge of Content and Teaching are principally Operation knowledge (connections mostly inside) for teachers.

Network analysis undoubtedly helped us to investigate how 364 relations between 136 knowledge topics were connected in the teacher knowledge network. The main findings are summarized in Table 12.
Table 12. Summary of findings reported in article III.

<table>
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<tr>
<td>Sections of teacher knowledge</td>
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<tr>
<td>• Strongly connected parts of teacher knowledge network refers to the sections of teacher knowledge.</td>
</tr>
<tr>
<td>• The sections of teacher knowledge are identified with the aids of network analysis.</td>
</tr>
<tr>
<td>• Since the sections are based on how knowledge is perceived, they can describe not only what kind of knowledge is needed for teaching but also where and why the knowledge is needed.</td>
</tr>
<tr>
<td>Findings</td>
</tr>
<tr>
<td>• Based on the group of student teachers, eleven sections of teacher knowledge were identified.</td>
</tr>
<tr>
<td>• Most of the sections are connected with two key sections, which indicates that selecting appropriate teaching methods and promoting different ways to learn mathematics are largely central questions of teacher knowledge.</td>
</tr>
<tr>
<td>• The sections are, however, mismatched with domain definitions of MKT, which indicates that the six MKT domains are connected to each other within the sections.</td>
</tr>
<tr>
<td>• The findings suggest that the domains of CCK and KCC have mostly external connections, while the domains of HCK and SCK have in- and out-connections in balance. and the domains of KCS and KCT have mostly internal connections.</td>
</tr>
<tr>
<td>• The findings indicate that the knowledge referred to six domains of MKT, which exist in a hierarchical sequence in the minds of student teachers.</td>
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In our opinion, the main findings summarized in Table 12 demonstrate that a network analysis may offer a new perspective in current research into teacher knowledge.
5 Discussion

In what follows, initially attention will be paid to the research aims and the main findings, culminating in a consideration of the trustworthiness of the research as a whole, and concluding with a presentation of the final conclusions and their implications, including future research scenarios.

5.1 RESEARCH AIMS AND FINDINGS

The survey study reported in Articles I and II was designed and implemented as a means of examining teacher educators’ and graduated teachers’ perceptions of the contents and their perceived development in the mathematics teacher education program offered at the University of Eastern Finland. Approximately half of all the teachers who have graduated from the teacher education program in the last decade and about eighty percent of all the teacher educators who were currently educating future teachers participated in the study. A framework drawing on Mathematical Knowledge for Teaching (MKT) was used in the design of the survey and also in the analysis of the data in order to construct a frame concerning potential development needs (e.g. Ball et al., 2008). The findings suggest development needs exist that are related to both the mathematical and the pedagogical knowledge of future teachers.

The first two development needs were related to graduated teachers’ Common Content Knowledge. In understanding how university mathematics prepares students for school teaching, the graduated teachers experienced the following difficulties: 1) university mathematics does not prepare students for teaching all of the topics of school mathematics; and 2) university mathematics is excessively academic and too complex, and there
are insufficient linkages between university and school mathematics.

The graduated teachers noted that there were mathematical topics, such as statistics and geometry, that are a compulsory part of the school curriculum, but university mathematics did not prepare sufficiently for teaching such topics (Article I). The teacher educators and the graduated teachers’ shared the perception that graduated teachers have only a modest knowledge of statistics and geometry compared to other mathematics topics (Article II). Some of the graduated teachers recalled that, following graduation, they needed to study less emphasized topics such as statistics and geometry before they were ready to teach them themselves (Article I). However, the mathematics majors felt that they possessed a better knowledge of geometry than did mathematics minors (Article II). In the mathematics teacher education program studied in this project, the student teachers were not offered statistics, while geometry was offered, but, because it was an optional course, only mathematics majors usually studied took the course (Article II). While some of the graduated teachers had found that statistics and geometry were not dealt with at all, university mathematics was nevertheless generally considered to be too academic, complex, or in-depth (Article I). The graduated teachers argued that in some respects they remained unable to understand how university mathematics was related to school mathematics (Article I). They claimed that if there were no appropriate connections between university and school mathematics, then university mathematics was pointless for them.

Mathematics is usually regarded differently at school and university level. School mathematics places its main emphasis on calculating and applying mathematics, and hence school mathematics is a more likely tool for understanding and facilitating everyday life. In contrast, university mathematics places its own main emphasis on proving theorems and it has an axiomatic-deductive structure, and hence university mathematics is more conceptual and abstract (Dreher, Lindmeier,
& Heinze, 2016, p. 220). In consequence, a student may experience difficulty in understanding the nature of university mathematics, especially at the start of their studies (Ottinger, Kollar, & Ufer 2016). Understanding the interrelations between school mathematics and university mathematics in both top-down and bottom-up directions may in fact be a new type of knowledge for teachers, and hence this knowledge may be needed so that the links between school and university mathematics can be understood (Dreher et al., 2016).

In the present study, the graduated teachers suggested that some course contents received too much attention or that they were excessively in-depth in university mathematics in comparison to school mathematics (Article I). Although the teacher educators considered that studying too much mathematics was unlikely to be harmful for teachers, the graduated teachers tended to be disappointed because they had studied areas of mathematics for which they have had no use in school teaching. In this case, the question is more about how much pure mathematical knowledge related to different topics do graduated teachers in fact need? Monk (1994) found that a relationship exists between the number of university mathematics courses completed by school teachers and their students’ actual achievement in mathematics. The relationship was found to be curvilinear, which means that precisely five university calculus courses provided the “optimal usefulness” appreciated by teachers themselves (Monk, 1994). Subsequently, Monk and King (1994) found more evidence of a cumulative relationship between the university courses taken by teachers and their students’ achievements in mathematics. It is a self-evident truth that mathematics teachers need “some” knowledge of mathematics, but there is only vague agreement on how much they need (Schmidt, Houang, & Cogan, 2011). However, finding an answer to the question *How much pure mathematical knowledge do mathematics teachers need?* is important for mathematics teacher education programs precisely because the extent to which mathematics is emphasized in mathematics teacher education has an impact on teachers’ knowledge, and, in turn, teachers’ knowledge has its own impact on students’
achievements (Schmidt, Houang, & Cogan, 2011). These findings suggest that mathematics teacher education programs should possibly explore the differences in the ways in which various mathematical topics are emphasized in teacher education. Qian and Youngs (2016) have also found that the content of mathematics courses has a more powerful effect on future teachers’ mathematical content knowledge than the number of courses mastered. These findings suggest that perhaps the most relevant question should be not about how much but what kind of contents teacher education programs should offer. In other words, it should be about considering how the fixed time period available may be used most effectively when educating future teachers.

Our findings indicated that the six MKT domains might be emphasized differently in the teacher education program that we examined. Both the graduated teachers and the teacher educators were of the opinion that the graduated teachers had all learned their mathematical and pedagogical knowledge in a rather similar way, and the differences occurred in the six MKT domains. They also felt that the graduated teachers had learned the topics included in Common Content Knowledge better than they had the topics included in Specialized Content Knowledge. The results indicate that the pure mathematics contents are possibly emphasized more than the contents specially designed for teachers. Further evidence in support of this interpretation included the fact that the graduated teachers argued that this particular education program was biased too much toward one-sided mathematical contents for their own needs. They also expressed the criticism that the mathematical contents were designed for future mathematicians rather than future teachers. They felt that too many of the mathematical courses focused excessively on proving and presenting theorems, an approach that might be appropriate for future mathematician but rather less so for the teaching profession.

This criticism relates to more widely expressed concern that appropriate subject matter knowledge may well be different for
teachers and mathematicians. Fletcher (as cited in Dreher et al., 2016, p. 219) stated in 1975 that “The mathematics teacher requires a general knowledge of mathematics in order to be able to communicate with other mathematicians and also to establish his credentials; but he also requires special knowledge of certain areas of mathematics, in the way that an engineer or an astronomer requires special knowledge. […]. It is part of our problem that the teacher’s special mathematical knowledge is inadequately defined and insufficiently esteemed.” Ball, et al (2008) identified and defined this type of knowledge as Specialized Content Knowledge. Knowing, for example, how to subtract is Common Content Knowledge but being able to evaluate alternative algorithms for subtracting is Specialized Content Knowledge, and hence “Specialized Content Knowledge goes beyond knowing how and it includes also knowing why” (Copur-Gencturk & Lubienski, 2013, p. 213).

In the present study, the graduated teachers recalled that the mathematics teacher education program included a course named School mathematics, which they had found important and useful. They said that the course contents were the same as for actual school mathematics, and the implementation of the course resembled the mathematics teaching conducted in schools. As a solution to both problems, i.e. that current mathematics courses do not take account of 1) school mathematics, and 2) the special characteristics of the teaching profession, the graduated teachers suggested that contents of mathematics course should be designed separately: for future teachers and for mathematicians. The teacher educators also suggested that their courses would not work properly for future teachers, and so they suggested that the contents of the two courses should be at least partly differentiated, for future mathematicians and for future teachers. Doing so would be broadly in line with the well-known recommendations made by other mathematics educators (e.g., Ball et al., 2008; O’Meara, 2011).

It is well known that to achieve effective teaching, teachers also need pedagogical content knowledge (e.g. Shulman, 1986; Ball et al., 2008). Pedagogical content knowledge concerns, for example,
understanding the characteristics of teaching and learning mathematics (Ball et al., 2008). In the present study, the graduated teachers claimed that they had faced a number of problems in their classrooms, such as coping with students of divergent ability. They explained that some of their students struggled to learn mathematics. These students may harbor misconceptions or experience learning difficulties but they also had insufficient skills to be able to recognize the characteristics of those challenges. Furthermore, they had also noted that students have different mathematical abilities, but as experienced they did not know how such students should be supported. They suggested that one option might be to establish small groups based on the students’ differences and then differentiate their teaching to teach them more effectively. But the question remained of how such groups were to be selected? Could the selection be based on students’ prior mathematical knowledge, on their abilities, or on differences in the ways in which the students learn mathematics? The teachers participating in the study suggested that teacher education did not offer enough tools for them to be able to differentiate their teaching in this way and hence this question remained open. In other words, they still lack sufficient knowledge to be able to conduct heterogeneous classes.

When teachers are confronted with students of different abilities, they need to have a variety of kinds of knowledge at their disposal. In order to recognize the abilities and difficulties of their students, they require Knowledge of Content and Students, while to reach decisions concerning how to actually teach such a range of students, they need Knowledge of Content and Teaching (Ball et al., 2008). Teachers must hear and respond whenever students discover new aspects of mathematics, as well as knowing when a teacher should ask a question in order to elaborate on the students’ thinking, or when a teacher should introduce a new task for their students, as an extension of what they have learned previously (Ball et al., 2008). It seems reasonable to integrate Knowledge of Content and Students and Knowledge of Content and Teaching in situations where teachers can observe their students’ behavior and actions on the basis of their own
observations. Ball et al. (2008) also pointed out that in real classroom situations the six MKT domains should not regarded as separate from each other but rather as integrated. Recognizing a wrong answer is Common Content Knowledge, whereas identifying the nature of the error, especially if the error is unfamiliar, requires a good understanding of the special characteristics of mathematics, while an awareness of the common errors made in mathematics falls within the category of Knowledge of Content and Students (Ball et al., 2008).

In the present study, some of the graduated teachers had found that after graduation they knew mathematics and pedagogy well enough, but they still had difficulty in putting what they knew together and into practice (Article I). Thus, it may indeed be the case that, since teachers need a variety of different kinds of knowledge in actual teaching situations, connecting what they know in appropriate ways could be the real challenge. It is entirely possible that teachers possess sufficient mathematical and pedagogical knowledge but they nevertheless lack the skills that would enable them to connect the various types of knowledge appropriately. Shulman (1986) suggested that it was therefore important to study teachers’ own perceptions of the knowledge required in actual teaching in order to understand the complexities of teacher knowledge per se. According to Shulman (1986) this should indeed be investigated: “What are the domains and categories of content knowledge in the minds of teachers? How, for example, are content knowledge and general pedagogical knowledge related? In which forms are the domains and categories of knowledge represented in the minds of teachers?” (p. 9). Thus, the problem of how teachers themselves view the knowledge required for teaching, and the structure of the relationships between the types of knowledge involved, encouraged us to construct the following study.

In the study that followed, which was reported in Article III, student teachers wrote an essay under the rubric of “The kind of knowledge required for teaching mathematics”. In our analysis, we focused on the kind of knowledge required for teaching and how
the knowledge issues mentioned above are related to each other. By coding the knowledge topics, classifying them into the domains of MKT, and tracking how the topics are interrelated, we transformed eighteen essays into a large network and used network analysis tools to examine the interconnections of teacher knowledge.

Based on our analysis, we identified 136 knowledge topics and 364 connections between them. Interestingly, however, these connections were not located randomly in the network but, for the most part, they existed in a structural order. The knowledge topics related to Common Content Knowledge (CCK) and Knowledge of Content and Curriculum (KCC) mostly had only external outside connections, which indicates that they may be more like the background knowledge for other domains. The domains of Specialized Content Knowledge (SCK) and Horizontal Content Knowledge (HCK), for their part, possessed a balance of both internal and external connections, which suggests that these knowledge types require some awareness of background knowledge but they are also knowledge specifically about something. The domains of Knowledge of Content and Students (KCS) and Knowledge of Content and Teaching (KCT) mostly possessed internal connections, that these knowledge types require knowledge from all of the other four domains and also that they were mostly related to actions in the classroom.

The results is reasonable because the framework has been defined as Mathematical Knowledge for Teaching. To understand the various specific aspects of mathematics (SCK) and structural features of mathematics (HCK), teachers need first to acquire some pure mathematical knowledge (CCK) and curricular knowledge (KCC). Then, to understand the multifold aspects of learning mathematics (KCS) and teaching mathematics (KCT), teachers should have an understanding of specific aspects of mathematics (SCK) and of the structural features of mathematics (HCK), but then teachers also need pure mathematical knowledge (CCK) and curricular knowledge (KCC). Our findings suggest that CCK and KCC are possibly more like foundation
knowledge, while HCK and SCK are more like transformation knowledge, and KCS and KCT are more like operation knowledge for teachers. Our findings provide some empirical evidence that suggests that teachers may need to regard the knowledge required in teaching in a logical sequence.

The results bear some similarities with the frameworks referred to as *The ladder of knowledge* (O’Meara, 2011) and *Knowledge quartet* (Rowland et al., 2007). On the ladder of knowledge, the first step is subject matter knowledge, the second is pedagogical knowledge, and the last is knowledge of effective teaching. The ladder of knowledge presents a strong argument in favor of the structural nature of teacher knowledge. O’Meara (2011, p. 31) suggested that “Teachers must combine their knowledge of general mathematical concepts and applications as well as their pedagogical knowledge and convert it into representations, explanations and analogies that students will understand and appreciate”. In the Knowledge quartet, mathematical knowledge and beliefs are referred to as Foundation knowledge, while the other three domains are based on it. Rowland et al. (2007) argue that the nature of foundation knowledge is different because it is something that people hold no matter whether they use it relevantly or not. In addition, other researchers have suggested that there is a logical sequence in the actual process of acquiring the knowledge required for teaching. For example, Wu (2005, p. 6) suggests that learning pedagogical content knowledge without first learning mathematical knowledge is the same as asking teachers to run before they can walk.

In addition to using network analysis methods to investigate the structural features of teacher knowledge, they were also used for to find what kind of knowledge topics were strongly interconnected (Article III). Network analysis methods were used to classify the types of knowledge into groups based on how strongly they were interconnected. In the context of teacher knowledge, we call these strongly connected parts of the network *Sections of teacher knowledge*. In the present study, eleven different sections of teacher knowledge were identified. However, none of...
them completely matched the domain definitions of MKT. The reason why the sections differ from the MKT domains is related to the way in which the sections were constructed. Unlike the MKT domains, these sections describe not only what kind of knowledge is required for teaching mathematics but also why and where this knowledge is required. It can therefore be argued that a practice-based perspective is included in these sections, but at the same time these sections may effectively describe how teacher knowledge is constructed in the minds of the study group.

In general, the teacher knowledge sections are constructed based on the connections in the minds of teachers, and hence they describe effectively how teacher knowledge is seen in practice. In consequence, it can well be claimed that calling the sections practice-based is highly apposite. In contrast, Ball et al. (2008) referred to MKT as practice-based theory although the six MKT domains have been formulated by also taking Shulman’s (1986) framework into account, and hence the six MKT domains cannot be purely practice-based.

5.2 EVALUATING THE RESEARCH WORK

Many aspects are related to the effectiveness of mathematics teacher education, for example, social status of mathematics, selectivity of teacher education, goals of teacher education, course contents, the effectiveness of instruction, teacher educators’ and future teachers’ knowledge and beliefs (e.g. Tatto et al., 2008; Hsieh et al., 2011). Teacher knowledge is one but significant perspective on both mathematics teacher education and teachers’ professional competence (see Schmidt, Houang, & Cogan, 2011). Therefore, the aim of this dissertation was investigating the current state and development needs of mathematics teacher education at the University of Eastern Finland through MKT framework.

This study contained two main phases. The first was mixed methods research, in which the qualitative part was reported in
Article I and the quantitative part in Article II. The second phase was then the qualitative research reported in Article III. In evaluating mixed methods research, many researchers have recommended using standard separated evaluation procedures for quantitative and qualitative approaches (e.g. Creswell, 2009, p. 219; Onwuegbuzie & Johnson, 2006, p. 56). In mixed-methods research, however, the threats to the validity and reliability of the quantitative and qualitative phases can be also synthesized (Ihantola & Kihn, 2011; Onwuegbuzie & Johnson, 2006). The analysis published by Ihantola and Kihn (2011) indicates that qualitative, quantitative, and mixed approaches yield valuable analogies in terms of research validity and reliability. For example, terms such as internal and external validity and reliability can be used in quantitative research, analogously to contextual validity, generalizability, and transferability, while procedural reliability can be used in the context of qualitative research, and the terms internal (contextual) validity, external validity (generalizability and transferability), and (procedural) reliability can be used in the context of mixed methods research (see Ihantola & Kihn, 2011). Ihantola and Kihn (2011) argued that the legitimation framework provided by Onwuegbuzie and Johnson (2006) was nevertheless extensive and appeared to be the most promising approach for addressing the potential threats to internal validity in mixed methods research.

In what follows, therefore, a discussion will be presented of how reliability and external validity threats are taken into account in the present study. Then a consideration of the application of mixed methods from the viewpoints associated with the framework of internal validity and especially legitimation is made. The main features of these perspectives have been previously presented in studies by Onwuegbuzie and Johnson (2006) and by Ihantola and Kihn (2011).

5.2.1 Reliability

Reliability refers to the quality of measurements and to the repeatability and consistency of research instruments used
Reliability focuses in the main on the research process and its measurement, and responds to the question of what has been measured and what was intended (Ihantola & Kihn, 2011). In quantitative research, reliability refers to the consistency of variables or set of variables, while lack of reliability usually refers to random or chance errors. In consequence, pre-testing, using clear standard instructions, avoiding abstract concepts, sequencing questions adequately, and determining the length of the survey should all be considered (Ihantola & Kihn, 2011).

In order to produce a comprehensible survey in the present research, four independent researchers examined and improved the wording of the instructions, statements, and questions. In order to construct a fluent survey, the sequence of the questions and the length of the survey were considered, and the survey was also piloted by two people from within the study group and two who were outside the group. The two people outside the study group were used because they were able to evaluate the intelligibility of the wording and the abstractness of the concepts used. All of the eight individuals who participated the pre-testing and piloting provided their opinions about the intelligibility of the instructions, statements and questions, and made suggestions concerning how the survey could be developed. The final version of the survey was then produced based on the pre-testing and piloting.

The Cronbach Alpha test is widely used in testing the consistency of survey instruments. Higher Cronbach alpha scores denote the existence of a stronger correlation between the items tested (Cronbach, 1951; Tavakol & Dennick, 2011). Since the items surveyed were designed in such a way that the statements could be related to the different MKT domains, the consistency of the statements was tested within these categories. The Cronbach alpha scores were .931 for subject matter knowledge items and .950 for pedagogical content knowledge items. These Cronbach Alpha scores are relatively high, which suggests that the survey instrument was consistent and that the possibility
of random or chance error was minimal (see Tavakol & Dennick, 2011). However, as a result of the low number of teacher educators’ responses, only the practicing teachers’ survey responses could be tested. Although the instrument may be consistent, it is possible that the survey instrument may not be stable, which means that consecutive tests may produce different results. These consecutive tests may nevertheless improve the stability of the particular survey instrument.

We also investigated the similarity of the teacher educators’ and the practicing teachers’ perceptions. Teachers’ and students’ perceptions can be interconnected in many different ways (see Trigwell, Prosser, & Waterhouse, 1999), but it is not self-evident that teachers evaluate their students’ learning in the same way as the students themselves. In consequence, we investigated the extent to which the views expressed by the teacher educators and practicing teachers about the graduated teachers’ learning were similar. Mean values were calculated based on the perceptions of all of the practicing teachers and teacher educators. The Pearson correlation between these views was 0.502, with a strong significance (p<.001), which indicates that the teacher educators and the graduated teachers shared relatively similar perceptions. We consider that since the different viewpoints suggest similar conclusions, the reliability of these findings has been enhanced.

In qualitative research, reliability refers to capturing a phenomenon “accurately” (Ihantola & Kihn, 2011). The ultimate question here was whether another person would be able to examine the work and reach similar conclusions (Ihantola & Kihn, 2011). Furthermore, Ihantola and Kihn (2011, p. 44) describe how “errors may also occur in data classification, attaching data to constructs, drawing linkages between constructs, reduction, interpretation and development of links with theory, etc.”. For example, inaccurate and/or unsystematic interview questions and/or

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1 Alpha score 0.931 in subject matter area gives 0.13 error variance (random error) and Alpha score 0.95 in pedagogical content knowledge gives 0.10 error variance (see Tavakol & Dennick, 2011).
transcriptions, a failure to take notes, or data collected over long periods may all be threats to the reliability of qualitative research.

We made use of two recognized approaches to analyze qualitative data in the present study: conventional content analysis and direct content analysis (Hsieh & Shannon, 2005). In conventional content analysis, the categories are derived directly from data, while direct content analysis starts from theory and its categories are used as guidance in the analysis (Hsieh & Shannon, 2005).

In the first research phase, we used the MKT domains as guidance in sorting into categories the various challenges related to the contents of mathematics teacher education (Article I). This analysis was similar to that conducted in the study made by Markworth et al. (2009), where they coded their interview responses and conversational topics on the basis of the domains of MKT. In the second phase of the research, we looked in detail at the student teachers’ perceptions of the knowledge required for teaching (Article III). For this, data obtained from essays was analyzed in a similar way to that used in the first research phase: the same issues were selected into data-based categories (conventional content analysis) and data-based categories produced were classified in terms of the domains of MKT (direct content analysis). On this occasion, however, the relationships between the knowledge topics mentioned were also tracked and the data was transformed into Gephi format for further analysis.

With the aid of Gephi, we next identified the various sections of teacher knowledge. In Gephi, modularity counting and the Atlas2 algorithm can be used to identify the tightly interconnected parts of the network under study (Jacomy et al., 2014; Khokhar, 2015). We made use of both in studying the robustness of the network and in locating, from our own perspective, the various sections of teacher knowledge. There is a wide agreement that various algorithms can be used for “community detecting”, but there is less agreement about what is the most rapid and accurate algorithm (e.g. Zhou, Wang, & Wang,
2012; Chen, Kuzmin, & Szymanski, 2014; Wang, Chen, & Lu, 2007). The study by Traag, Krings and Van Dooren (2013) suggests that new methods for calculating “significance” may be needed in order to determine how “good” the detected communities in fact are. From this perspective, when the tools of network analysis are used to investigate the sections of teacher knowledge, various algorithms may be needed to confirm the sections identified.

5.2.2 External validity

External validity refers to the generalizability and transferability of research findings. In quantitative research, this involves generalizing the results by applying them to other samples, time periods, and settings (Ihantola & Kihn, 2011). Population validity refers to the question of whether such interference is valid only for the selected sample or also, rather, for a larger population. For example, if the sample is not random or if the sample size is too small, this may constitute a serious threat (Ihantola & Kihn, 2011). Time validity refers to threats that consist of findings that may be confined only to that particular period of time; ideally, the results should be generalizable to other time periods. Environmental validity refers to issues that consist of findings that can be generalized across settings. In qualitative research, external validity refers to connecting empirical findings to theories, thus explaining how new evidence enhances understanding of research questions and how findings may be transferable (Ihantola & Kihn, 2011).

In the first phase of the research, our survey targeted the teacher educators who were actively working in the program in 2012 and also the teachers who had graduated during the period 2002-2012. Firstly, examining these study groups was undoubtedly relevant to the task of gathering information about mathematics teacher education. Secondly, the whole population was examined, in which the sample sizes were comprehensive and the backgrounds of the participants in the samples correlated with the population as a whole. It should also be noted that the
perspectives included in the survey questions aimed at capturing the respondents’ personal experiences and perceptions of the mathematics teacher education program under study. The mathematics majors considered that they had, for example, learned geometry and the history of mathematics better than the mathematics minors (Article II). A possible explanation for this view may be that the majors usually take optional courses concerned with geometry and the history of mathematics, whereas the minors rarely did so. Thus, it seems logical that circumstances, environment, and contents in mathematics teacher education bore some relation to the respondents’ perceptions. However, if we consider the generalizability and transferability of these research findings, our first suspicion is about the environmental validity. The teacher educators and practicing teachers who participated came from only one teacher education program, whereas we were examining respondents’ perceptions across the six MKT domains, with numerous challenges coinciding with those found in previous research findings. Our analysis of the challenges by means of a theoretical lens found that our own findings agreed with previous research findings, which in turn demonstrated that similar challenges have been detected in a range of different countries. Despite this, our findings describe the primary circumstances of the program under study, and it is likely that that many similar challenges exist in mathematics teacher education in many parts of the world. In consequence, however, it must also be recognized that the challenges revealed by individual teacher education programs will need to be identified so that a broader picture can be constructed of this multifaceted phenomenon.

The time validity of these findings depends on the stability of the program. If the contents are developed within the mathematics teacher education program under study, the results may no longer be relevant. As Black and Wiliam (2004) pointed out, information gained from such an assessment may always be used to develop the field. If information about the challenges reported in Articles I and II is used as the basis for development of the field, similar concerns may no longer be an obstacle.
In the second phase of the project, the student teachers’ perceptions of the knowledge required for teaching were collected with the aid of essays. All of the eighteen participants were from the same university, taking the same degree course, and all of them were generally at the same stage in the final year of their masters studies. The student teachers’ perceptions of the knowledge required for teaching mathematics were transformed into a large network and the network was studied in detail. Our findings reveal how all of the eighteen student teachers viewed teacher knowledge as a group. But can we draw further conclusions from this study group? The answer appears to be in both the affirmative and the negative. Although the population is limited and the context somewhat limited in terms of further conclusions about teacher knowledge itself, the network analysis contains some valuable new information related to the respondents’ views about the knowledge required for teaching. The findings suggest that the student teachers viewed knowledge and its function logically, which means that the sections of teacher knowledge describing not only what kind of knowledge is required but also where or why. Furthermore, as a result of the various connections, the structural features can be also related to teacher knowledge. Examining the sequences and structures of teacher knowledge may reveal something new about teacher knowledge. Hence, we consider that this study has to a large extent been a pioneer in the field, and while the results in the raw cannot be generalized, the approach integrating a network into research into teacher knowledge can be applied internationally.

5.2.3 Internal validity

Ihantola and Kihn (2011) showed that in quantitative research internal validity is concerned, for example, with considering how theory has been constructed on the basis of previous studies, on appropriate use of statistical control variables, on the logic of research and existing theory, and on drawing valid conclusions from a study. On the other hand, in qualitative research the internal validity depends, for example, on capturing authentically lived experiences and/or on studying whole
phenomenon, and also on demonstrating that the researcher has fully understood the case under study (Ihantola & Kihn, 2011).

In the present study, our first concern was the suitability of MKT as a framework. In consequence, our first requirement has been to discuss to some extent the origins and evolution of MKT itself. Its origins reside in elementary mathematics teaching, since its framework has been constructed from large and longitudinal documentations of third grade mathematics teaching (Ball et al., 2008). This qualitative data has included a year’s worth of video recordings of teaching, audiotapes of classroom lessons, transcripts, copies of students’ written classwork, homework, and quizzes, as well as teachers plans, notes, and reflections (Ball et al., 2008). Analysis of this data in turn generated hypothetical characterizations of MKT (e.g. Ball, Hill, & Bass, 2005; Ball et al., 2008). Hill, Schilling, and Ball (2004) subsequently developed specific measurements of MKT that could be used to test this hypothetical characterization. The measurements were validated and tested against practise (Hill, Blunk, Charalambous et al., 2008) and against students’ achievements (Hill et al., 2005). At a later point in time, these tested and validated measurements have been widely distributed globally, in e.g. Ireland (Delaney et al., 2008), South Korea (Kwon et al., 2012), Ghana (Cole, 2012), Indonesia (Ng, 2012), Norway (Fauskanger et al., 2012), Iceland (Jóhannsdóttir & Gísladóttir, 2014), and Malawi (Kazima et al., 2016). A review study by Hoover et al. (2016) indicates that in the field of teacher knowledge there are no similar survey instruments that have become standardized and equally popular.

Even MKT is grounded in elementary mathematics teaching, it does not simply describe elementary teachers’ knowledge. Firstly, MKT has theoretical foundations, and it strongly agrees with other teacher knowledge frameworks (e.g. Ernest, 1989; Fennema & Franke, 1992; O’Meara, 2011; Rowland et al., 2009). Secondly, choosing to use MKT as the basis for our research is justified since numerous similarities exist between the six MKT domains and in the aims of Finnish mathematics teacher education (see Article II, pp. 29-30). However, further research
will be needed to investigate how, for example, Specialized Content Knowledge is different for teachers working at different levels. This idea has already been discussed by a number of researchers. Dreher et al. (2016) and Speer, King and Howell (2014) suggest that understanding the relationship between Common Content Knowledge and Specialized Content Knowledge for teachers at different school-levels requires further investigation.

5.2.4 Legitimation

Ihantola and Kihn (2011, p.51) have pointed out that “[…] even if the quality of the quantitative and qualitative parts of the research is excellent, problems may still occur in validating the meta-inferences of mixed methods research”. In addressing this issue, Ihantola and Kihn (2011) suggested that the threats of mixed methods research can be evaluated within a framework of legitimation. The legitimation framework contains nine perspectives for evaluating the validity of mixed methods research (see Onwuegbuzie & Johnson, 2006, p. 57). Finally, however, attention will be paid here to how well we have been able in this study to integrate its qualitative and quantitative approaches.

*Sample integration* refers to a relationship between samples in quantitative and qualitative research (Onwuegbuzie & Johnson, 2006). In mixed methods research, there may be a situation where only a part of a quantitative study of people is also subject to qualitative interview. However, in this case, the individuals interviewed are selected from a larger study group, which may result in the quantitative and qualitative findings are mismatching. Ihantola and Kihn (2011) suggested that this problem could be avoided if exactly the same individuals are involved in both the quantitative and qualitative phases of mixed methods research. In the present study, a survey was used to collect qualitative and quantitative data, and all of the qualitative and quantitative findings were based on exactly the same group of teacher educators and graduated teachers. Hence, sample
integration is not a threat in this study since exactly the same individuals were studied both qualitatively and quantitatively.

*Inside-Outside* legitimation refers to the research problem that occurs when quantitative and qualitative viewpoints are not in balance (Ihantola & Kihn, 2011). Normally, a quantitative approach will offer outsiders’ perceptions and qualitative research will seek interpretations produced by insiders (Ihantola & Kihn, 2011). Onwuegbuzie and Johnson (2006, p. 58) state that “The ability to do this can be compromised when a researcher is ethnocentric or, on the other hand, when a researcher becomes so involved with the group that he or she “goes native””. The use of peer-reviewing, member-checking, or participant-reviewing are strategies for avoiding threats to inside-outside legitimation (Onwuegbuzie & Johnson, 2006). In the present study, one of the researchers has his/her own experience of studying on and graduating from the mathematics teacher education program under study. Furthermore, all three of the other researchers are actively working as teacher educators within the same program. In consequence, we can say that, to a greater or lesser degree, all four researchers unavoidably have insiders’ perceptions of the program under study. In the present case, however, this prior experience did not constitute a threat since it helped us to understand and locate the challenges that arose in this context. In sum, both the qualitative and the quantitative data types were threatened similarly, both data types were analyzed separately in their entirety, and both studies were evaluated by outsiders through the peer-review process.

*Weakness minimization* means using the strengths of either the quantitative or the qualitative approach to compensate for the weakness of the other (Onwuegbuzie & Johnson, 2006). Ihantola and Kihn (2011) suggest that identifying such threats is important for mixed methods researchers. An epistemologically quantitative approach is more deductive and works better if prior studies or theories exist, while on the other hand a qualitative approach is more inductive and works in support of such situations as when we are observing reality to achieve new
knowledge. There are many responses involving the framework of teacher knowledge to the question of what kind of knowledge is required in teaching mathematics, and hence there is research-based knowledge concerning the kind of aims that mathematics teacher education should include. However, the development needs of mathematics teacher education may also concern something other than teacher knowledge, and hence it is a broader phenomenon that is also affected by contextual concerns and personal experiences. Thus, from our perspective, a quantitative approach was more suitable for measuring how respondents regarded the learning undergone by the graduated teachers, while a qualitative approach was useful for collecting information about the potential developmental requirements of mathematics teacher education. In sum, therefore, we can state that weakness minimization was taken into consideration in the research design when appropriate approaches were undergoing selection.

Sequential legitimation refers to the appropriate use of mixed methods sequentially. There are various ways in which mixed methods designs can be used sequentially (see Creswell, & Plano Clark, 2011, p. 109). For example, quantitative findings may either guide qualitative research work or not. In the case of sequential legitimation, the question is then of whether changing the selected order of the qualitative and quantitative phases make any difference to the results and interpretations. In the present study, sequential legitimation can be ignored because both data types were collected concurrently.

Conversion is a legitimation type that refers to the quantification of qualitative data, or vice versa (Onwuegbuzie & Johnson, 2006). This may become a threat, for example, when numbers are used to interpret the qualitative findings (Ihantola & Kihn, 2011). Onwuegbuzie and Johnson (2006, p. 58) suggest that for a qualitative researcher, “counting can provide useful information about how often or how many or how much”, but it may also be a threat. For example, when a mixed methods researcher interviews only a restricted number of the individuals under
study but then states percentages related to ways in which the group as a whole considered something as if this would match the perceptions of the whole group (Ihantola & Kihn, 2011). In the qualitative phase, we grouped the challenges that had emerged in the MKT domains and listed them in a sequence based on their frequency. All of the perceptions and challenges that arose were presented, but the challenges that occurred with a high frequency were likely to be more discussed than those with lesser support. In other words, the challenges that were repeated more frequently in the theory-based categories may have received a greater emphasis in Article I. In this case, numbers were used to interpret the qualitative data, and so this was not a threat since precisely the same individuals participated in the two studies. In other words, study groups were the same in both the qualitative and the quantitative phases, and thus the percentage values also matched. In fact, this was the way in which we were able to treat the qualitative and quantitative data similarly, since the quantitative data was interpreted according to its mean values, which also emphasizes the averages rather than the extremities in the respondents’ perceptions.

Paradigmatic mixing is a legitimation type that is related to the combination of approaches that contain different paradigmatic assumptions (Onwuegbuzie & Johnson, 2006). Ihantola and Kihn, (2011) suggested that “Combining the approaches can be problematic because of competing dualisms of paradigmatic assumptions: epistemological (objectivist vs subjectivist), ontological (single reality vs multiple realities), axiological (value free vs value bound), methodological (deductive logic vs inductive logic), and rhetorical (formal vs informal writing style) assumptions” (p. 49). In the present study, we investigated teacher educators’ and graduated teachers’ perceptions of the developmental requirements of mathematics teacher education. In ontological terms, we have been investigating what is essentially a single reality. Both of the study groups had been in contact with the same mathematics teacher education program and they regarded this mathematics teacher education from rather similar perspectives. The survey contained specific statements and also open questions about the knowledge
required in teaching mathematics. The six MKT domains were used as guidance in the formulation of the statements, and hence they contained a hypothesis concerning the kind of knowledge that is required for teaching mathematics. However, the statements were formulated based on our knowledge of MKT, and hence the approach is not truly deductive or value free. In contrast, the open-ended questions did not contain this hypothesis, but the responses were analyzed using the six domains of MKT as guidance. From these perspectives, neither our qualitative nor our quantitative approaches could, in epistemological terms, be purely objectivist or subjectivist, nor could they, in methodological terms, be purely inductive or deductive. Rather, both approaches relied on aspects that lay between the two. From this perspective, in both approaches our ontological, epistemological, methodological, axiological, and rhetorical beliefs were quite similar. However, since MKT was also built into both approaches, it helped us in combining and integrating the qualitative and quantitative data types and their respective findings.

Commensurability is a legitimation type that refers to the challenge posed in seeing a third viewpoint that is neither quantitative nor qualitative but mixed. Onwuegbuzie and Johnson (2006) have suggested that “In order to meet this type of legitimation, the mixed researcher must learn to make Gestalt switches from a qualitative lens to a quantitative lens, going back and forth, again and again […] Through an iterative process, a third viewpoint is created, a viewpoint that is informed by, is separate from, and goes beyond what is provided by either a pure qualitative viewpoint or a pure quantitative viewpoint” (p. 59). Frequently, we noted that wider perceptive on quantitative data indicated that teacher educators and graduated teachers both found that the six areas of teacher knowledge are learned differently. This in turn indicated that there may very well be development needs in certain areas of teacher knowledge. However, since a quantitative approach could not provide an adequate answer to that question, qualitative data were also investigated. Qualitative data indicate that some developmental needs undoubtedly exist, but this then initiates the need to
examine the similarities between the quantitative and qualitative data, and hence the quantitative data had to be re-investigated, this time more closely. Systematic analysis of statements confirmed that there were similarities between the developmental needs and poorly learned topics. Commensurability legitimation was taken into account by working back and forth through the quantitative and qualitative data and comparing both data types to construct a third perspective.

Multiple validities refer to seeking meta-inference in the results provided by the qualitative and quantitative data (Onwuegbuzie & Johnson, 2006). Integrating the findings of both approaches offers a greater validity for mixed methods research. In the present study, we have combined the findings of the qualitative and quantitative approaches and we have also investigated the similarities between the perceptions of teacher educators and practicing teachers. By triangulating the quantitative and qualitative findings, we thus have taken into account the multiple validities.

Political legitimation refers to the tensions or conflicts that may emerge when ideologically based qualitative and quantitative researchers collaborate in a mixed methods study (Ihantola & Kihn, 2011, p. 50). Onwuegbuzie and Johnson (2006) have suggested that “These tensions include any value or ideologically based conflicts that occur when different researchers are used for the quantitative and qualitative phases of a study, as well as differences in perspectives about contradictions and paradoxes that arise when the quantitative and qualitative findings are compared and contrasted” (p. 59). Four different researchers have participated in the present study, but none of them has been ideologically orientated or has favored any specific approaches, and hence it can be claimed that the political legitimation of this study can be ignored.
5.3 CONCLUSION AND IMPLICATIONS

Mathematicians and their students generally regard mathematics differently, and learners may encounter numerous pitfalls before they can acquire a mathematician’s perception of how mathematics can be seen as a single, accurate construction made up of mathematical concepts. Tall (1991, p. 17) illustrates this perception with an example in which mathematics is like a jigsaw puzzle. In this example, the mathematician observes mathematics from the experts’ viewpoint, and therefore she/he is able to see how all of the pieces of mathematics are connected and joined together perfectly as one. Students, however, observe mathematics from a novice’s perspective and therefore do not see all of the individual pieces of mathematics as similar, and then even glimpsing a picture of the totality that approximates to that of the experts may, for them, represent a true challenge. In our opinion, therefore, the reason why mathematics teacher education should be evaluated in particular in its entirety is analogous to this example. In consequence, this conclusion commences with an example analogous to a reconstructed jigsaw puzzle.

For in-service teachers, every course in mathematics teacher education is like a piece of a jigsaw puzzle. Each course contributes to an increase in knowledge and skills that are important in the teaching profession, and each puzzle piece has a particular meaning and place in the totality. But the student teacher may see the pieces as they are presented, in isolation, like separate pieces of a jigsaw puzzle for which no total picture is available. The actual scenario may in fact be worse. As the student teacher encounters each piece of the puzzle, she/he will form a personal conceptual image from the particular context that may be at variance with the formal idea. Thus, not only is no picture available for the puzzle, but the pieces themselves may now have different shapes so that they no longer fit together. In the worst scenario, the teacher educators themselves do not know what the individual how pieces should look like nor what the jigsaw puzzle should illustrate.
Based on our results, we propose that teacher education programs in all concerned countries should implement *curricular work* as support in the assistance for evaluation and development of mathematics teacher education programs. Teacher educators should, in concert, consider what the aims of single courses are, how the courses are connected, and what the total aims are of mathematics teacher education. In teacher education, each course has its own particular aims, meanings, and purposes in relation to the totality, but at the same time each teacher educator should have a similar vision of the common aims of the contents of mathematics teacher education. This may prove to be challenging especially if student teachers are educated by teacher educators who themselves possess to little knowledge of the demands of the teaching profession. Despite this, research-based knowledge can offer assistance in considering the demands of the teaching profession and in joining all of the pieces of the jigsaw puzzle together. However, our research understanding is still imperfect and cannot yet explain the total impact with regard to effective mathematics teacher education (Hsieh et al., 2011). Cooperation-based curricular work could help teacher educators to better perceive their roles in the totality of teacher education, and the written aims and strategies related to achieving ambitions will facilitate a sharpening of the stated aims, which in turn may help researchers and educators to evaluate and develop mathematics teacher education.

There is a need for the use of a theoretical lens in exploring the aims of mathematics teacher education. Mathematical Knowledge for Teaching (MKT), for example, provides six divergent perspectives on the knowledge required in teaching mathematics, and hence these viewpoints can be associated at least in part with the aims of mathematics teacher education. In addition, we have found that the challenges located in the contents of mathematics teacher education can also be observed by means of MKT viewpoints, and hence these challenges seem to become more explicit when the subject matter knowledge and pedagogical content knowledge are divided into six more detailed components.
Many viewpoints are needed so that further information about the present state of mathematics teacher education can be gathered. In the process of evaluating how the aims of mathematics teacher education can be achieved, focusing solely on teacher educators’ perceptions is insufficient (Hsieh et al., 2011). For this kind of evaluation, teacher educators’ perceptions are too limited, whereas graduated teachers have completed all of the relevant courses and thus they will have the most recent experience of working in the school classroom. Hence, asking them to evaluate the ways in which the contents of their studies have prepared them for their profession, what kind of challenges they have faced in the classroom, and how they would develop mathematics teacher education, would appear to be highly relevant. Investigating teacher educators’ perceptions is also important even if teacher educators cannot fully evaluate how the courses in mathematics teacher education fit together or how they prepare for the teaching profession.

In formulating a curriculum for mathematics teacher education, consideration should also be given to how its various components are emphasized. The time period for educating student teachers is fixed but also limited. Hence, consideration needs to be given, for example, to the extent to which pure mathematics is highlighted in comparison to mathematical knowledge that is particular to teachers, or to the ways in which the various aspects of teaching mathematics are emphasized in comparison with learning mathematics. Finding the balance between these emphases is relevant since the contents of mathematics teacher education have an impact on future teachers’ knowledge and also on students’ personal achievement in mathematics (Hill et al., 2005; Monk, 1994; Monk & King, 1994; Schmidt, Houang, & Cogan, 2011). The question should not, however, be related solely to how much but also to what kind of contents will be the most relevant for future teachers.

In addition, in formulating the curriculum for mathematics teacher education, consideration should be given to how all of the course contents work together, and how they form the total
picture represented by the jigsaw puzzle. It is more than likely that student teachers cannot see all of the pieces as they should, and for this reason they need to be challenged while receiving the idea of the jigsaw puzzle as a whole. In addition, perhaps the worst scenario – that even teacher educators do not know properly what their own roles and purposes are within the construction of the total picture – is not in fact particularly exaggerated. If this is indeed true, it is no surprise that, after graduation, teachers may see, for example, that there is a gap between university and school mathematics, or that mathematical knowledge and pedagogical knowledge are experienced as isolated types of knowledge with no proper connections. It is possible that the jigsaw puzzle has been designed weakly and that it lacks important ways in which the pieces can be joined together. In other words, perhaps the jigsaw puzzle was incomplete even in its opening stages. Hence, attempting to unravel the various factors involved in measuring the overall effectiveness of mathematics teacher education deserves more research attention (Hsieh et al., 2011). In addition, teacher educators’ and student teachers’ perceptions should be investigated regularly in order to observe potential inconsistencies.

A perhaps at least partial solution to this problem could be found if an investigation were to be made of teachers’ and teacher educators’ own perceptions of the knowledge that is relevant in teaching. There is already a strong theoretical context related to the kind of knowledge required for teaching mathematics (e.g. Ball et al., 2008; Ernest, 1989; Fennema & Franke, 1992; O’Meara, 2011; Rowland et al., 2009). However, less attention has yet been paid to examining teachers’ and teacher educators’ own perceptions of teacher knowledge and its diversity. Future research should focus more on the kind of knowledge required in teaching, especially in the minds of teachers and their educators, but even more so, there is a need to investigate the connections within teacher knowledge.
Research attention should be also paid to investigating the complexity of teachers’ own perceptions of teacher knowledge, in addition to examining the relationships between teachers’ subject matter knowledge and pedagogical content knowledge. Two different types of research have already started to investigate the relationships between teachers’ subject matter knowledge and pedagogical content knowledge. Large-scale studies most often use distinct test items to measure the relationship, while small-scale studies aim at unraveling the relationship between teachers’ subject matter knowledge and pedagogical content knowledge as enacted in the classroom (Depaepe et al., 2013).

We have demonstrated that a network analysis can offer a new – and possibly not previously studied – method of investigating relationships in teacher knowledge. As we have shown, taking account of how knowledge topics are connected in the minds of individuals or groups of individuals permits, for example, investigation of the structure of teacher knowledge. It is possible that teacher knowledge generally includes a hierarchy, which means that something has to be learned before new knowledge can be achieved (e.g. O’Meara, 2011; Wu, 2005). This kind of hierarchy is reasonable, but less research attention has thus far been paid to this particular research subject. Research findings of this kind could, for example, assist mathematics teacher education programs in organizing course contents in an appropriate sequence. Even if the jigsaw puzzle is perfectly designed, the ease with which the puzzle can be completed depends on the sequencing of the jigsaw pieces themselves.

Examining, for example, pathways, flows, clusters, and hubs of a teacher knowledge network can generate a new research understanding of teacher knowledge, which could in turn yield fresh solutions for educating future mathematics teachers more effectively. If the connections in the minds of respondents are taken account in the research, it allows the researcher to examine the kinds of sequences that teacher knowledge includes. Within a teacher knowledge network these sequences are analogous to storylines, which might in turn explain how mathematical or
pedagogical knowledge is integrated or how theoretical knowledge is transformed into practical knowledge. On the other hand, the absence of such storylines may perhaps be related to the theory-to-practice challenge, which in the present case may mean that knowledge is experienced as too theoretical while mathematical and pedagogical knowledge is experienced as isolated and not properly interconnected.

We have also demonstrated that the research into of teacher knowledge becomes somewhat more complex when network analysis is used to study how the various knowledge topics are related to each other in the minds of teachers. At the same time, however, this simplifies the research since there are no more than four different types of knowledge:

1. Foundation knowledge
2. Transformation knowledge
3. Operation knowledge
4. Isolated knowledge

In this model, all four knowledge types can be identified in the teacher knowledge network by observing the arrows associated with each node. Nodes that have only outward arrows consist of Foundation knowledge, nodes that have outward and also inward arrows consist of Transformation knowledge, nodes that have only inward arrows consist of Operation knowledge, while nodes that have no arrows at all consist of Isolated knowledge. We would predict that investigating these four knowledge types may reveal something new about the nature of teacher knowledge.

First, there is a need to examine how different study groups view/use the knowledge required in teaching and what such knowledge looks like if it is examined through these four knowledge types. It is possible that knowledge that is pointless can be identified as isolated knowledge. On the other hand, knowledge that is challenging perhaps requires a lot of background knowledge, or knowledge that is useful may have many different purposes. It seems to be reasonable that there can
be many different variations in the ways that individuals view knowledge and its relationships, although it is also possible that the purposes, meanings, and nature of teacher knowledge itself are more restricted. It is indeed possible that different study groups may even view the kinds of knowledge relevant to teaching and their interconnections differently; it is also possible that the aims and nature of such teacher knowledge may be viewed as parallel. Thus, the diverse relationships of teacher knowledge require further investigation that will help to describe the complexity of teacher knowledge in the minds of teachers.

The connections or lack of connections may also reveal how differently experts and novices view teacher knowledge. It is possible that teacher educators possess relatively strong theoretical knowledge, while, in some situations, seeing how knowledge is linked to practice may also be a challenge for the experts. On the other hand, some teachers are perhaps better at making connections and transforming theoretical knowledge into practice while they are teaching. This network approach could be used to produce various frameworks of teacher knowledge, based on investigations of different study groups. Each node and arrow can contain an almost unlimited amount of information, and therefore the nodes or arrows could be classified in several ways synchronously. In addition, this network approach could be used to compare, but also to integrate, a number of different frameworks. This kind of research might offer new answers about teacher knowledge, but integrating network analysis into the research of teacher knowledge might also help in the development of a new multidisciplinary framework of teacher knowledge.

This framework takes account how teacher knowledge is interlinked, and hence the framework can respond not only to what kind of knowledge is required for teaching mathematics but also to where and why this knowledge is required. Perspectives such as these increase substantially the usefulness of theoretical knowledge vis-à-vis its practical application. With this kind of knowledge, teacher educators could, for example, justify where
and why the contents of mathematics teacher education are needed in teaching mathematics or how a relevant knowledge related to teaching can be seen differently. If this framework is grounded in a large corpus of international data, the teacher knowledge network will also permit investigation of the contextual differences between a range of countries. That, in turn, may reveal how a teacher’s knowledge is integrated into planning decisions and especially into classroom events, which is yet another insufficiently researched aspect of teacher knowledge (Escudero & Sánchez, 2007).

In sum, we would encourage teacher educators internationally to launch curricular work in support of evaluating and developing their mathematics teacher education programs. Taking account of the perceptions and cooperation of teachers’ and their educators’ may itself provide keys in achieving more effective teacher education. In addition, we would also encourage other researchers to investigate the relationships and the nature of teacher knowledge, for example, by using network analysis methods that may yield innovative insights into teacher knowledge.
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Teachers and their Educators - Views on Contents and their Development Needs in Mathematics Teacher Education

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Teachers and their Educators – Views on Contents and their Development Needs in Mathematics Teacher Education

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Abstract: Finland has scored well in international assessments (e.g. PISA, TIMSS), and the pressure to attain excellent scores has activated a drive toward even more effective mathematics teacher education. This article presents the results of a qualitative assessment of the mathematics teacher education provided by the University of Eastern Finland. In this study, the views held by practicing teachers (N=101) and teacher educators (N=19) are compared so that the outstanding development needs of mathematics teacher education in terms of their contents can be revealed. The data was gathered via an electronic survey and was mainly analyzed using data-driven methods. In addition, framework provided by Mathematical Knowledge for Teaching (MKT) was used to categorize the respondents’ views regarding the contents of mathematics teacher education and to develop general guidelines for the reform of mathematics teacher education. The results indicate that mathematics teacher education should include pure mathematical content (Common Content Knowledge, CCK) and mathematical content that will have been designed only for future teachers (Specialized Content Knowledge, SCK). Teacher educators and practicing teachers both held the view that the relevance of CCK studies depend on the connections between university and school mathematics. Pedagogical studies should also be reformed because practicing teachers have realized that effective teaching (Knowledge of Content and Teaching, KCT) requires knowledge about learning mathematics (Knowledge of Content and Students, KCS) that is not offered in the current educational system on a sufficiently broad basis. In this study, suggestions for developing mathematics teacher education were mostly connected to four domains of MKT: (CCK, SCK, KCT and KCS). Interestingly those domains are the same domains which has been empirically tested and better conceptualized.

Keywords: Mathematical Knowledge for Teaching, MKT, mathematics teacher education, evaluating teacher education, contents of mathematics teacher education.

Introduction

Finland has scored well in international assessments (e.g., PISA, TIMSS), and the Finnish school system has been rated as being of top quality. Finnish teacher education has also been evaluated as high in quality from an international perspective (Kivirauma & Ruoho, 2007; Tryggvason, 2009). An important reason for this success is that Finnish teachers are educated both systematically and extensively, and every qualified teacher must have a Master’s degree (Tryggvason, 2009). It is claimed that Finnish teacher education is the result of a long-term, research-based development (Tryggvason, 2009). However, the voices of practicing mathematics teachers and teacher educators have not received attention enough in the research field. Are these two groups satisfied with the current contents of mathematics teacher education and what kind of needs of development they see at the moment?

In the present study we focus on practicing mathematics teachers’ and teacher educators’ views on mathematics teacher education. The practicing teachers participating in this study graduated in the period of 2002–2012 and they nowadays teach at school level, which enables them to evaluate the contents of teacher education from a perspective of the teacher’s profession. In addition, when the survey was implemented the teacher educators were actively working as teacher educators. We were interested in discovering how these two subject groups saw the present contents
of the Mathematics Teacher Education Program (MTEP) at the University of Eastern Finland and also in how they would develop the teacher education program. We sought answers to the following research questions:

1. How do teacher educators and practicing mathematics teachers regard the course contents of mathematics teacher education?
2. What kind of recommendations would teacher educators and practicing mathematics teachers make for improving mathematics teacher education program?

The views held by practicing teachers and teacher educators play an important role in developing teacher education. There may be a possibility that the contents are not regarded as being as useful as teacher educators assume. It is also possible that practicing teachers and teacher educators hold conflicting views about the contents. Hence, the views of both groups are important in order to be able to form a coherent picture of the current status of teacher education and to construct an extensive basis for the development work.

Our methodical aim has been to test a theoretical framework called Mathematical Knowledge for Teaching (MKT) (Ball, Thames & Phelps, 2008) through the process of categorizing practicing teachers’ and teacher educators’ views. This framework appeared to be promising for categorizing these views, since it has previously worked relatively well in classifying teacher knowledge (see Markworth, Goodwin, & Glisson, 2009; Fauskanger, Jakobsen, Mosvold, & Bjuland, 2012).

Conceptualizing the Teaching of Mathematics

Mathematical knowledge for teaching

There was an increasing interest in the 1980s in teacher qualifications and methods of effective teaching that would influence student learning. Lee Shulman proposed that a teacher also needs to possess other types of knowledge than pure subject matter knowledge in order to teach so that students would understand. In 1986 Lee Shulman introduced a new term, pedagogical content knowledge (PCK). According to Shulman (1986), teachers must have an integrated knowledge of subject and pedagogy, some kind of amalgam knowledge. Initially, Shulman considered PCK to be a topic-specific subcategory of content knowledge, which included two further subcategories: knowledge of representations and knowledge of learning difficulties and strategies for overcoming them. Shulman’s later model consisted of seven categories, of which PCK was one, with no subcategories (Shulman, 1987). By proposing PCK as one out of seven categories of conceptualization, Shulman neglected the potential for integration among these categories and the hierarchies that might exist between them, and left the task of further development of the concept to other researchers (Hashweh, 2005).

Shulman’s conceptualization has been criticized for its restricted and ambiguous definitions of categories (Ball et al., 2008, Hashweh, 2005). Ball et al. (2008) claim that the terms PCK and content knowledge are frequently confused with common pedagogical skills. Meredith (1995) argues that PCK as defined by Shulman simply implies one type of pedagogy rooted in particular representations of prior knowledge. Meredith suggests that learners have a built-in competence for constructing their own understanding of subject matter, but Shulman’s PCK seems not to encompass alternative views of teaching. Meredith argues that Shulman’s definition of PCK leads to teaching methods where the teacher will explain and illustrate procedures while learners practice the procedures by using examples. Thus, the teacher’s role can be seen as transmitting mathematical knowledge and helping learners to acquire understanding.

Shulman’s conceptualization has also been claimed to ignore the interaction between the different categories, assuming that knowledge is static rather than possessing a dynamic nature (Hashweh, 2005; Fennema & Franke, 1992). Fennema and Franke (1992) argue that teacher
knowledge frequently changes in light of classroom interaction experiences, and hence teachers’ beliefs should form an important part of the conceptualization. According to Fennema and Franke, teacher knowledge can be divided into four parts: knowledge of content, knowledge of pedagogy, knowledge of students’ cognitions, and teachers’ beliefs. At the center of this model is context specific knowledge, which can be seen as dynamic knowledge, since it occurs in the context of the classroom. In this model, PCK consists of teachers’ knowledge of teaching procedures, such as effective strategies for planning, classroom routines, behavior management techniques, classroom organization procedures, and motivational techniques. Fennema and Franke (1992) see teacher knowledge as interactive and dynamic in nature and they suggest that no single domain of teacher knowledge plays a particular role in the effective teaching of mathematics.

Rowland, Turner, Thwaites & Huckstep (2009) developed The Knowledge Quartet conceptualization, which was based on Shulman’s conceptualization (1986) with respect to Fenneman and Franke conceptualization (1992). The Knowledge Quartet was generated by categorizing elementary teachers’ classroom actions. The main aim of the research work was to investigate the relation between the teacher’s subject matter and PCK knowledge. Detailed analysis of the elementary mathematics lessons taught by pre-service teachers resulted in the identification of teacher knowledge framework. Rowland et al. (2009) suggest that the framework can be used to classify teachers’ actions in the context of a classroom.

One of the most promising recent efforts in discovering the kind of knowledge and skills that are needed for high-quality mathematics teaching has been the theoretical framework known as Mathematical knowledge for teaching (MKT), as posited by Ball and her associates¹. In this model, subject matter knowledge is categorized into three domains: common content knowledge (CCK), horizon content knowledge (HCK), and specialized content knowledge (SCK) (see Figure 1). In addition, PCK consists of three parts: knowledge of content and student (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC). The domains CCK, HCK, and SCK are subject matter knowledge that requires no knowledge concerning either the students or pedagogy. In addition, the domains of KCS, KCT, and KCC are the kind of knowledge that requires an integrated knowledge made up of subject matter knowledge and pedagogical knowledge (Sleep, 2009), as in Shulman’s (1986) conceptualization. According to Sleep (2009), four of the domains (CCK, SCK, KCS and KCT) have been empirically tested and better conceptualized, while two of the domains (HCK and KCC) are still in the earlier stages of conceptualization.

¹The University of Michigan projects Mathematics Teaching and Learning to Teach project (MTLT) and Learning Mathematics for Teaching project (LMT) produced plenty of details to form MKT, e.g., Hill & Ball, 2004; Hill, Schilling & Ball, 2004; Hill, Rowan & Ball, 2005; Hill & Lubienski, 2007; Hill, Ball, Sleep et al., 2007; Hill, 2007; Schilling, 2007; Schilling, Blunk & Hill, 2007; Schilling & Hill, 2007; Hill, Ball, Blunk, et al., 2007; Hill, Dean & Goffney, 2007; Hill, Ball & Schilling, 2008; Delaney, Ball, Hill et al., 2008; Stylianides & Ball, 2008; Hill, Blunk, Charalambous et al., 2008; Ball, Thames & Phelps, 2008; Ball & Forzani, 2009; Thames & Ball, 2010.
Figure 1. Domains of Mathematical Knowledge for Teaching (MKT) by Ball et al. (2008)

CCK consists of mathematical knowledge and skills used in any settings, including in settings other than teaching, and it includes calculating, solving problems, and other common mathematical knowledge that is not unique to teaching (Ball et al., 2008). SCK is mathematical knowledge and skills that are peculiar to teaching, and is typically not intended for other settings than teaching (Ball et al., 2008). In other words, SCK consists of the mathematical knowledge and skills that a mathematician does not need, while at the same time they are needed by a teacher in order to practice effective teaching. HCK consists of mathematical knowledge of the mathematical structures and also awareness of how mathematical topics are related to each other in a curriculum (Ball et al., 2008). This means that a teacher needs to know how topics are related to each other at different school levels and how mathematics is actually constructed.

KCS consists of amalgam knowledge of students, learning, and mathematics (Ball et al., 2008). A teacher must be able to anticipate students’ difficulties, hear and respond to students’ thinking, and choose suitable examples and presentations while teaching. A teacher’s action in planning and teaching requires awareness of students’ conceptions and misconceptions of different mathematical topics. KCT is also amalgam knowledge of teaching and mathematics (Ball et al. 2008). Teachers need KCT knowledge in choosing proper activities, exercises and representations for different topics. Teachers need KCT knowledge for both planning and teaching. One important part of this knowledge for teachers is to recognize situations where teachers should diverge from their original planning, for example, if a student makes a mathematical discovery.

KCC represents amalgam knowledge of mathematics and curriculum. According to Sleep (2009), a teacher needs to know the contents of the curriculum, but Ball et al. (2008) offer only a restricted definition of KCC and hence the kind of knowledge and skills that KCC includes remains unclear. Our preliminary analysis of the data in the present study showed that if MKT is used to organize practicing teachers’ and educators’ views, the KCC domain has to be modified. The practicing teachers and teacher educators mentioned skills and knowledge related to teaching equipment. Hence, our conceptualization states that KCC also includes knowledge and skills related to teaching materials (including textbooks, other materials, etc.), teaching instruments (blackboard, overhead projector, etc.), and technology (computer, smart board, calculators, software, etc.).
The Evolution of MKT

The development of MKT started with the study of classroom actions with a view to identifying the knowledge needed for teaching mathematics (Ball & Bass, 2003). This work continued with the formation of hypothetical characterizations of MKT (e.g. Ball, Hill & Bass, 2005; Ball et al., 2008). Thereafter, Hill, Schilling, and Ball (2004) developed specific measurements of MKT that could be used to test this hypothetical characterization. In the case of validating measurements, the Michigan group tested measurements against practice (Hill, Blunk, Charalambos, et al., 2008) and also against students’ achievements (Hill, Rowan & Ball, 2005). Thereafter, MKT has been used to develop the contents of teacher education in ways that should help teachers to acquire the knowledge required for teaching mathematics (Ball, Sleep, Boerst & Bass, 2009).

Markworth, Goodwin and Glisson (2009) have used MKT to evaluate what student teachers have learned during a teaching practicum course. They coded interview responses and conversational topics on the basis of the domains of MKT. By using MKT in their analysis, Markworth, Goodwin and Glisson (2009) were able to capture more detailed information about the subject matter knowledge and pedagogical content knowledge that student teachers had gained during the teaching practicum course.

In the course of this study, practicing teachers and teacher educators suggested various recommendations for improving the mathematics teacher education program. To identify these suggestions systematically, we used MKT in a similar way to that of Markworth, Goodwin and Glisson (2009). This meant that suggested recommendations for improving mathematics teacher education could be classified in terms of six domains of MKT.

Method

Context

This study was implemented at the University of Eastern Finland, which offers two programs for students of mathematics: one for mathematicians and another for teachers. The programs are almost identical in their respective amounts of mathematics courses, but they differ in minor subjects. In the present study, we concentrate on the program for teachers, Mathematics Teacher Education Program, MTEP.

MTEP includes a Bachelor’s degree (180 cp\(^2\)) and a Master’s degree (120 cp). Both degrees are required for a student to qualify as a mathematics teacher in Finland. MTEP includes mathematical studies (130 cp), pedagogical studies (60 cp), and studies in one or two minor subjects (60 cp each). Most mathematical studies are traditional mathematics courses, which are compulsory for both future teachers and mathematicians (e.g. calculus, analysis, algebra, differential equations, etc.).

The pedagogical studies include theoretical studies focusing on teaching and learning (30 cp), the didactics of mathematics (10 cp), and teaching practice (20 cp). Teaching and learning courses are intended to all subject teachers and courses are concerning teaching and learning in general. However, the following courses which are intended only to forthcoming subject teachers of mathematics enables taking into account the special aspects of mathematics. Teaching practice is undertaken at the university teacher training school. Student teachers plan their own teaching sequences or lessons under the guidance of a subject teacher. Student teachers’ lessons are evaluated and feedback is also provided. The amount of student teaching is approximately 50

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\(^2\) One credit point (cp) is the equivalent of 25 hours of study. The recommendation is to complete 60 cp of studies per year.
lessons. The training school teachers’ task is to guide student teachers in addition to performing their own ordinary teaching work.

Student teachers can choose to study any school subject as a minor subject, but the most typical choices are physics or chemistry, or both. In its entirety, MTEP provides students with the competence to teach mathematics and minor subjects at lower or upper secondary schools and vocational schools.

Sample

The data was collected in the course of two separate electronic surveys conducted in 2012–2013. The first survey was aimed at mathematics teachers who had graduated from the UEF during the period 2002–2012. Our sample (N=101) includes 54% of all teachers who have graduated from UEF during period 2002–2012. In the sample, the majors taken by our respondents were 72% (73) mathematics, 20% (20) physics and 8% (8) chemistry, which makes the sample similar to the distribution of graduated teachers according to their major subject. All of the respondents, with the exception of one, had had previous experience of teaching mathematics at school or they were working as teachers when the survey was implemented.

The second survey targeted the teacher educators in mathematics at the UEF, who taught either mathematical studies or pedagogical studies or were guiding teaching practice. Our sample (N=19) includes 79% of all of the teacher educators in mathematics at the UEF. In the sample, 74% (14) of the teacher educators taught mathematics and 26% (5) worked in pedagogical studies or teaching practice. To conceal the respondents’ identities, the teacher educators in the fields of both pedagogical studies and teaching practice were placed in the same category.

Instrument and data analysis

The study was implemented with the aid of a survey that included statements about the knowledge and skills learned in MTEP and open questions about the present state and the future of MTEP.

The survey conducted with practicing teachers included three open questions about MTEP.
1. Evaluate the contents of mathematical studies in MTEP, especially with regard to the work of a mathematics teacher.
2. Evaluate the contents of the pedagogical studies and teaching practice in MTEP, especially with regard to the work of a mathematics teacher.
3. Suggestions regarding the development of mathematics teacher education would also be appreciated.

The survey involving the UEF teacher educators included two open questions.
4. Evaluate the contents of the studies you teach, especially with regard to the work of a mathematics teacher.
5. Please make a suggestion regarding the development of mathematics teacher education would also be appreciated.

The data was analyzed using qualitative content analysis (Tesch, 1990; Hickey & Kipping, 1996; Mayring, 2000; Hsieh & Shannon, 2005). Hsieh and Shannon (2005) have identified three different approaches to qualitative content analysis that can be used to interpret meaning from the content of text data: conventional, directed or summative (Hsieh & Shannon, 2005).

Our analysis started with reading the data several times to achieve immersion and obtain a sense of the whole (Tesch, 1990). Then, practicing teachers’ and teacher educators’ perceptions about the contents of MTEP (Questions 1, 2 and 4) were analyzed with *Conventional Content Analysis* (Hsieh & Shannon, 2005). In the conventional content analysis, coding categories are
derived directly from the text data. In our data, respondents’ personal experience or more like attitudes towards contents emerged clearly from data. Each respondent was placed in one of these categories (Figure 2).

A majority of the practicing teachers’ mentioned only issues that should be developed in the contents of MTEP (Questions 1 and 2), and therefore their responses were placed in the category of *In need of development*. The contents of this category were analyzed with directed content analysis, which is a more structured process than the conventional approach (Hickey & Kipping, 1996; Hsieh & Shannon, 2005). Direct content analysis starts with a theory, which is used for coding text data (Hsieh & Shannon, 2005). Generally, a goal of the directed approach is to validate or conceptually extend a theoretical framework or theory (Hsieh & Shannon, 2005). *Mathematical Knowledge for Teaching (MKT)* framework was a starting point for designing the survey, and each statement was designed to be interconnected to the domain of MKT. In the planning, we noticed a possibility for using MKT for directed content analysis. A pre-analysis of the data indicated that all the issues in question 1 and many of the issues in question 2 can be categorized with MKT. Issues beyond MKT were categorized with the conventional content analysis in case of question 2.

Both the surveys also included blank spaces for other suggestions for the development of the teacher education program (Questions 3 and 5). Many of the respondents did, however, mention the same issues which they already mentioned in the previous question related to the contents. Therefore we used directed content analysis similarly as in the categorization of the suggestions related to the six domains of MKT. Suggestions beyond MKT were categorized with the conventional content analysis. Previous questions in the survey covered the majority of respondents’ ideas, and so there were only a few new ideas among these suggestions.
Figure 2. Text data analysis was performed with conventional and direct content analysis (Hsieh & Shannon, 2005).

Results

The results of the study are presented in two parts. First, we discuss how teacher educators and practicing mathematics teachers view the contents of mathematics teacher education. Second, our discussion focuses on teacher educators’ and practicing mathematics teachers’ ideas for developing teacher education. Suggestions for developing mathematics teacher education will be represented in tables where the categories have been provided mainly by MKT.

Views on the contents of mathematics teacher education

Practicing teachers’ views concerning the contents of mathematics studies. The categorization of practicing mathematics teachers’ views concerning the contents of mathematical showed that one fifth of the respondents (21%) viewed the contents neutrally. Half of them considered the number of mathematics courses appropriate for teachers, while the other half gave no reasons for their responses. A small minority of the respondents (7%) did not consider the contents of the mathematics courses useful for teachers. In most cases, the reason for this was that the courses were considered to provide too complex a discussion of mathematics in comparison with the mathematics needed in a teacher’s work. No fully positive views appeared in the categorized responses.

A majority of the practicing teachers (59%) provided only suggestions related to developing the present contents of mathematics courses. These suggestions were analyzed again by using MKT. Most of these suggestions (79%) were related to improving student teachers’ subject matter knowledge, while one fifth of them were concerned with developing student teachers’ pedagogical knowledge and skills (see Table 1).
Table 1. Categorization of practicing teachers’ (N=60) suggestions for developing the content of mathematical studies. Each respondent was permitted to mention more than one issue.

<table>
<thead>
<tr>
<th>Category</th>
<th>Domain of MKT</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Common content knowledge (CCK)</strong></td>
<td>Present course contents are not linked with school mathematics</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>Present course contents are not the same as in schools</td>
<td>11</td>
</tr>
<tr>
<td></td>
<td>More geometry</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>More financial and statistical mathematics</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Wider knowledge of mathematical concepts</td>
<td>2</td>
</tr>
<tr>
<td><strong>Specialized content knowledge (SCK)</strong></td>
<td>Present mathematical studies should be separate for student teachers and mathematicians</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>More school mathematics needed</td>
<td>12</td>
</tr>
<tr>
<td><strong>Horizon content knowledge (HCK)</strong></td>
<td>Present course contents are not linked with each other</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>More skills concerned with teaching students at different levels</td>
<td>1</td>
</tr>
<tr>
<td><strong>Knowledge of content and students (KCS)</strong></td>
<td>More studies concerning learning difficulties in mathematics</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>More skills concerned with teaching students at different levels</td>
<td>3</td>
</tr>
<tr>
<td><strong>Knowledge of content and teaching (KCT)</strong></td>
<td>More courses about didactic mathematics</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>More courses about how to differentiate teaching</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>More skills to motivate students in mathematics</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>More studies about teaching problem solving</td>
<td>1</td>
</tr>
<tr>
<td><strong>Knowledge of content and curriculum (KCC)</strong></td>
<td>More courses about using technology in teaching mathematics</td>
<td>1</td>
</tr>
</tbody>
</table>

Common content knowledge (CCK). The practicing teachers mentioned that the present contents of mathematical studies are not the same as the contents of school mathematics. They were disappointed that they had studied so much mathematics that they had never used in their school teaching. The practicing teachers also mentioned that the present contents did not link properly with school mathematics. Some teachers claimed that presentations at university were either symbolic and theoretical or too complex in comparison with school mathematics, and hence it was hard to see how the course contents were linked with school mathematics. The practicing teachers mentioned that they lacked the competence to teach geometry and financial or statistical mathematics, and so they suggested that MTEP should include more courses in those domains. It is evident that practicing teachers need mathematical content knowledge (Ball et al., 2008), but the opinions of the practicing teachers indicated that to be useful for future teachers, the contents should be linked to school mathematics.

Specialized content knowledge (SCK). The practicing teachers suggested that mathematical studies should be arranged differently for future teachers and for mathematicians. They argued that the current integrated mathematical courses do not support future teachers properly. Some practicing teachers said that at university the focus of mathematics was proving and presenting results, whereas school mathematics consisted of rather more than that. Almost all the respondents suggested that the mathematical representations of the course contents should be modified with respect to teachers’ actual work. According to them, in the current situation MTEP includes too much pure mathematics and not enough school mathematics. Most of them recalled that in MTEP there was a course called School mathematics, which they found important and useful. The contents of the course were the same as in actual school mathematics, and the implementation of the course resembled the mathematics teaching conducted in schools. In consequence, they argued that they learned the contents of such courses well and that they had been able to use the course contents in their teaching work. All of them argued that there should be more courses of this kind in MTEP. All
of these practicing teachers’ views are linked to the definition of SCK (Ball et al., 2008): practicing teachers need mathematical knowledge that is particular to the needs of teachers.

**Horizontal content knowledge (HCK).** The practicing teachers argued that the contents of university mathematics courses were not interconnected or that the links could not be detected during the courses. In their view, courses that were in fact extensions of each other (e.g., calculus 1, calculus 2) were separate courses; alternatively, they were unable to detect the ways in which new mathematical concepts could be constructed on the basis of previously learned concepts. In the view of the respondents, the mathematical knowledge base ought to resemble a network, while, for them, the contents of MTEP did not support the construction of that kind of concept. Mathematical knowledge lacking a proper understanding of the structure of mathematics can be identified as the major challenge in the domain of HCK (see Ball & Bass, 2009).

**Knowledge of content and students (KCS).** The practicing teachers claimed that pedagogical issues can also be discussed during a mathematics course. They mentioned that they did not develop any clear idea of how students were actually learning mathematics during the mathematics courses. Some practicing teachers mentioned that they had too little competence in the issues concerned with mathematical learning difficulties. Some teachers also argued that they need more skills related to teaching mathematics to both weak and talented students at the same time.

**Knowledge of content and teaching (KCT).** The practicing teachers argued that issues concerned with teaching mathematics can also be handled in mathematical studies. They mentioned that didactic mathematics and studies about how to differentiate teaching should be included in mathematical studies.

**Knowledge of content and curriculum (KCC).** One practicing teacher argued that future mathematics teachers needed more enhanced skills concerned with the use of technology in teaching mathematics since teachers were increasingly using technology in schools.

**Practicing teachers’ views about the contents of pedagogical studies and teaching practice.** The categorization of practicing mathematics teachers’ views about the contents of pedagogical studies and teaching practice demonstrated that the respondents’ views were diverse. Small minorities of the respondents viewed these studies positively (2%) as useful for teachers; or neutrally (9%), often without providing reasons; or negatively (7%), considering the courses useless for teachers; but most of them (67%) consider that there was a need for development in these studies. 3 The categorization of their suggestions is presented in Table 2. These suggestions mainly concerned pedagogical content knowledge and skills (51%), and development of the structure of mathematics teacher education (40%).

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3 12% of respondents did not answer this question, and the responses of 3% were irrelevant.
Table 2. Categorization of practicing teachers’ (N=68) views on how to develop the content of pedagogical studies and teaching practice. Each respondent was permitted to mention more than one issue.

<table>
<thead>
<tr>
<th>Category</th>
<th>Domain of MKT</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pedagogical content knowledge and skills</td>
<td>Knowledge of content and students (KCS)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• More studies of learning difficulties in mathematics</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>• More skills concerned with how to handle students at different levels</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>• More skills concerned with evaluating students’ knowledge and skills</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>• More studies concerned with other learning theories</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Knowledge of content and teaching (KCT)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• More studies of teaching mathematics; didactic mathematics</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>• More training in the planning and teaching of complete courses</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>• More studies of how to differentiate teaching</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>• More skills concerned with motivating students of mathematics</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>• More courses about functional teaching methods, teaching problem-solving, or visualizing mathematics</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>• More studies of how to link learning theories to practice</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Knowledge of content and curriculum (KCC)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• More skills and knowledge to produce teaching materials of their own</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>• More studies of using technology in teaching mathematics</td>
<td>3</td>
</tr>
<tr>
<td>Structure of mathematics teacher education</td>
<td>Amount of studies</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• More teaching practice</td>
<td>21</td>
</tr>
<tr>
<td></td>
<td>• More pedagogical studies</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>• Compulsory update education after some years of teaching</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>• More studies of how to teach minor subjects</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Quality of studies</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Linking theory to practice</td>
<td>25</td>
</tr>
<tr>
<td></td>
<td>• Educators of practice teachers should give more advice about didactic issues</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Developing curriculum of MTEP</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Contents of pedagogical studies and practice should be better integrated</td>
<td>1</td>
</tr>
<tr>
<td>General issues</td>
<td>The other knowledge and skills</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• More studies of teachers’ extramural duties</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Common issues</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Departments’ cooperation should be improved</td>
<td>1</td>
</tr>
</tbody>
</table>

Knowledge of content and students (KCS). The practicing teachers suggested that the pedagogical studies and teaching practice should include more courses about the learning difficulties encountered in mathematics. Some teachers said that they were struggling with students who probably had learning difficulties and hence they needed more skills in order to be able to recognize and handle such students. Some of the practicing teachers also mentioned that the skills concerned with teaching students at different levels would be useful for them because student groups were often very heterogeneous. The practicing teachers also mentioned that they needed more knowledge and skills for evaluating student learning and more knowledge concerned with various learning theories, since students learn in different ways.

Knowledge of content and teaching (KCT). The practicing teachers demanded more skills for teaching mathematics. Some of them mentioned that the courses in didactic mathematics were useful for them, but that they needed more knowledge of this kind. The practicing teachers argued that their studies did not include enough courses on planning and teaching complete courses. They stressed that planning was the first thing that new teachers needed to undertake after graduation. The practicing teachers also mentioned that more skills for differentiating teaching and increasing student motivation would be of assistance.

Knowledge of content and curriculum (KCC). The practicing teachers said that textbooks or other printed material did not always fit their ideas about teaching, and so they would need more
skills to design their own teaching materials. The practicing teachers also mentioned that they needed more knowledge and skills concerned with using technology in teaching mathematics in a pedagogically reasonable way.

The number of courses. The practicing teachers said that both the teaching practice and the pedagogical studies were very useful and suggested that their number should be increased in MTEP. They felt that the teaching practice was a good place for trying out new teaching methods or for trying to transform pedagogical knowledge into practice. They also told about the use of useful and functional teaching methods learnt during their teaching practice in their actual work. Some of them mentioned encountering similar situations in the classrooms to those that had been discussed in the pedagogical studies, which had helped them to better understand the relevance of the pedagogical studies.

The quality of courses. The practicing teachers argued that the courses in the pedagogical studies and even courses about teaching and learning were too theoretical, which made linking theory with practice difficult. They described a feeling of learning a lot during these studies, but without having the necessary skills to apply this knowledge in classroom situations. Some teachers even felt that the pedagogical studies were useless because they had too few links with real-life teaching situations.

Other knowledge and skills. Some practicing teachers said that they were surprised by the duties that teachers had outside the classroom. They suggested that these issues should be discussed in mathematics teacher education.

Teacher educators’ views on the contents of their own courses. One fourth of the teacher educators (26%) viewed their own courses positively and considered that the courses were useful for future teachers (see Table 3). Many of them (42%) viewed their own courses neutrally and regarded the courses as having been only partly useful. Some teacher educators (16%) viewed their own courses negatively and indicated problems in the contents of courses that made them not very useful for teachers.
Table 3. Teacher educators’ (N=19) views about their courses and their suitability for future mathematics teachers. ME = teacher educator in mathematical studies, PTE = teacher educator in pedagogical studies and teaching practice.

<table>
<thead>
<tr>
<th>Class and justification</th>
<th>ME (N=14)</th>
<th>PTE (N=5)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive 26% (5)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Contents increase pure mathematical knowledge and the teaching methods used teach how to teach mathematics</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>• Contents are the same as in school mathematics</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>• No justification</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>Neutral 42% (8)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Contents are not the same as in school mathematics, but studies develop mathematical thinking</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>• Only some parts of contents link with school mathematics</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>• Some contents go beyond school mathematics or general knowledge for teachers</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>• Courses are non-compulsory for student teachers and therefore their contents are not useful for teachers</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Negative 16% (3)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Some courses are simply all-around education for teachers and in some courses there is not enough time to teach important issues</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>• Students’ knowledge is poor at the beginning of courses and therefore they cannot learn the contents</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>• Teachers learn contents but they have insufficient skills for using this knowledge in school teaching</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>Empty 16% (3)</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Positive (5). There were three mathematics educators (MEs) and two pedagogical studies and teaching practice educators (PTEs) who considered that the contents of their own course were useful for teachers. The PTEs did not justify their views, but the MEs argued that the contents of their courses increased student teachers’ mathematical knowledge and the teaching methods modeled the way to teach mathematics. The MEs underlined the significance of presenting things; they argued that it was important for student learning that the educator demonstrated how things worked. One ME argued that the contents of his courses were the same as in school mathematics and hence the contents were useful for future teachers.

Neutral (8). Seven MEs and one PTE considered the contents of their courses only partly useful for future teachers. One PT and two MEs claimed that, despite the contents not being the same as for school mathematics, the courses nevertheless developed students’ mathematical thinking, which was also important for future teachers. Two MEs claimed that only some parts of the contents were linked with school mathematics and therefore these parts were useful for future teachers. Another ME thought that the contents are “good to know”, but were unnecessary for teachers. Two MEs argued that the contents of their courses offered teachers only general knowledge since the contents were not specialized for use by teachers or the contents went beyond school mathematics. Both of them justified their views with the argument that future teachers needed a wide knowledge base in mathematics.

Negative (3). Two MEs and one PTE argued that the contents of their courses did not fully support future mathematics teachers. One ME claimed that in some courses s/he had too little time to teach issues that were important for teachers, while in other courses the contents were simply general knowledge for teachers. Another mathematics ME claimed that teachers usually learned the contents of his/her courses, but the course nevertheless did not provide them with the competence to apply this knowledge in their own teaching. One PE claimed that students had acquired insufficient earlier knowledge to learn the contents of his/her courses.
Teacher educators’ and practicing teachers’ suggestions for developing mathematics teacher education

The second research question was concerned with how practicing mathematics teachers and mathematics teacher educators would develop mathematics teacher education.

The practicing mathematics teachers made numerous suggestions for developing mathematics teacher education. The categorization of the suggestions in Table 4 shows that it would be valuable to develop teacher education both at the general level and also in terms of supporting future mathematics teachers’ subject matter knowledge and pedagogical knowledge and skills. More than half of the suggestions (60%) concerned the contents of teacher education that could be categorized with MKT. One third of the suggestions (28%) focused on the quality of the teaching or the quantity of the studies that were categorized as ideas for developing teacher education program. A minority of the suggestions (12%) concerned a number of general issues related to teacher education.

Suggestions for improving the contents of teacher education mostly concerned pedagogical knowledge and skills. Practicing teachers suggested that they would add courses about learning difficulties in mathematics, the evaluation of students’ mathematical know-how, and how to teach students with different levels of mathematical knowledge and skills. The practicing teachers also hoped that differentiating mathematics teaching would be discussed during teacher education, since classroom situations required that kind of competence from a teacher. They also suggested that the learning theories courses should be modified so that they would become easily applicable to one’s own teaching. The practicing teachers would also add future teachers’ knowledge and skills related to using technology in teaching mathematics because technology was assuming a more important role both in the classrooms and in society. Almost all of the suggestions concerning subject matter knowledge dealt with separate mathematics studies programs for future mathematicians and teachers. One common argument was that future teachers needed a different kind of mathematical knowledge from that used by mathematicians.

The ideas that were presented regarding development of the teacher education program concerned both the quality of teaching and the quantity of studies. The practicing teachers thought that the quality of teacher education could be increased by improving students’ learning. This could be achieved by modifying present teaching methods as well applying new interactive teaching methods that would include discussions. The practicing teachers also argued that the studies should be modified to be less theoretical because they felt that the present studies were too theoretical, causing the students problems in understanding the course contents to any depth. The practicing teachers also thought that the teaching practice supported their teacher growth. However, they considered that the length of the teaching practice could be increased.
Table 4. Categorization of practicing mathematics teachers’ (N=101) suggestions for developing mathematics teacher education. Each respondent was permitted to mention more than one issue.

<table>
<thead>
<tr>
<th>Category</th>
<th>Suggestions for improving education</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Knowledge of contents and students (KCS)</td>
<td>Courses about learning difficulties in mathematics</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>How to test students’ knowledge and skills</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>How to teach students at different stages of learning</td>
<td>2</td>
</tr>
<tr>
<td>Knowledge of contents and teaching (KCT)</td>
<td>How to differentiate teaching</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>How to bridge the gap between learning theories and practice</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>How to produce and use one’s own teaching materials</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Functional learning methods</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Learner-centered teaching methods</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Special education in mathematics</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>How to plan and teach complete courses</td>
<td>1</td>
</tr>
<tr>
<td>Knowledge of contents and curriculum (KCC)</td>
<td>How to use technology in teaching mathematics</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>Curricular knowledge</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Teaching methods based on technology</td>
<td>1</td>
</tr>
<tr>
<td>Subject matter knowledge and skills</td>
<td>Separate mathematics courses for teachers and mathematicians</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>Problem-solving</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Mathematics in different professions</td>
<td>1</td>
</tr>
<tr>
<td>Specialized content knowledge (SCK)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Horizon content knowledge (HCK)</td>
<td>Mathematical concepts at different school levels</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>The structure of mathematics</td>
<td>1</td>
</tr>
<tr>
<td>Ideas for developing the teacher education program</td>
<td>New teaching methods (e.g., more discussion)</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>Less theory – more practice – linking theory and practice</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>Integrating lectures and doing exercises in mathematical courses</td>
<td>1</td>
</tr>
<tr>
<td>Quality of teaching</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quantity of studies</td>
<td>More teaching practice</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>More pedagogical courses</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>More teaching practice and less pedagogical studies</td>
<td>1</td>
</tr>
<tr>
<td>General issues</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Improving cooperation</td>
<td>Cooperation between different departments</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Cooperation between university and schools</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Cooperation between students and educators; paying attention to students’ suggestions regarding development</td>
<td>1</td>
</tr>
<tr>
<td>Special suggestions</td>
<td>Teachers should be specialized in teaching at different school levels</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Subject teachers’ major should be in education</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Compulsory updating of education after some years of work experience</td>
<td>1</td>
</tr>
<tr>
<td>Beyond MKT knowledge</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>More knowledge about teachers’ duties out of class</td>
<td>2</td>
</tr>
<tr>
<td>Uncategorized responses</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Irrelevant</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>Blank</td>
<td>27</td>
</tr>
</tbody>
</table>

The teacher educators saw less reason for development than did the practicing teachers. Most of the teacher educators suggested developing mathematics teacher education by improving student teachers’ subject matter studies (see Table 5).
Table 5. Categorization of teacher educators’ (N=19) suggestions for developing mathematics teacher education.

<table>
<thead>
<tr>
<th>Category</th>
<th>Suggestion for improving education</th>
<th>f</th>
</tr>
</thead>
<tbody>
<tr>
<td>Subject matter knowledge and skills</td>
<td>Specialized content knowledge (SCK)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Contents of mathematics courses should be revised to be useful for future teachers</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>• Mathematics courses should be separately designed for teachers and mathematicians</td>
<td>3</td>
</tr>
<tr>
<td>Common content knowledge (CCK)</td>
<td>• More courses in mathematics</td>
<td>2</td>
</tr>
<tr>
<td>Horizon content knowledge (HCK)</td>
<td>• New course on the structures of mathematics</td>
<td>1</td>
</tr>
<tr>
<td>Developing the teacher education program</td>
<td>Updating structure of studies</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Combining mathematics and pedagogies courses as an integrated unit</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>• Re-scheduling courses in mathematics, pedagogies, and teaching practice</td>
<td>1</td>
</tr>
<tr>
<td>Uncategorized responses</td>
<td>• Irrelevant</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>• Blank</td>
<td>5</td>
</tr>
</tbody>
</table>

Most of the suggestions concerned modifying future teachers’ mathematical studies. Some educators suggested that the contents of present courses should be revised from the viewpoint of teacher’s work and current school curricula. Many of the respondents would develop courses to increase future teachers’ specialized content knowledge (SCK). Some educators also suggested that mathematics studies should be separately designed for future teachers and mathematicians, an idea that was also put forward by the practicing teachers. However, a few teacher educators argued that pure mathematics was the basis of good teaching and therefore the quantity of pure mathematics studies should be increased. One educator suggested that there was a need for developing a mathematics course whose rationale would be to link together the various domains of mathematics. We categorized this as an example of improving students’ horizon content knowledge.

Two teacher educators considered that the structure of the mathematics teacher education should be updated. Another educator suggested that mathematical and pedagogical studies should not be organized separately, since their separation prevented the possibility of linking theory and practice. Another teacher educator argued that studies should be better scheduled to help student teachers to acquire an integrated knowledge of subject matter and pedagogy. It should also be noted that a third of the respondents provided no suggestions for developing mathematics teacher education, and hence it remains unknown whether these respondents were satisfied with the current teacher education or not.

Discussion

This study has investigated teacher educators’ and practicing mathematics teachers’ views of the contents and the development needs of mathematics teacher education as provided by the University of Eastern Finland. Practicing teachers and teacher educators made various recommendations for improving mathematics teacher education program. We consider that we have been able to identify systematically and in a detailed way the kind of subject matter knowledge and pedagogical content knowledge that these recommendations concern by classifying them in terms of the domains of MKT. Challenges concerning the content of mathematics teacher education seem to become more explicit when subject matter knowledge and pedagogical content knowledge are divided into more detailed components. Markworth, Goodwin and Glisson (2009) found similar benefits when they used MKT to evaluate a single course in mathematics teacher education. The combined results show that a majority of the recommendations concerning the issues that will need to be examined in mathematics teacher education are closely related to four domains of Mathematical Knowledge for Teaching (CCK, SCK, KCS, and KCT). Interestingly, these four
domains have been more empirically tested and better conceptualized than the other two domains (Sleep, 2009).

Our results indicate that the majority of practicing mathematics teachers do not regard the present contents of mathematics studies to be fully functional for future mathematics teachers. The practicing teachers suggested, for instance, separate courses for future mathematics teachers and mathematicians, and the possibility of taking school mathematics into account in the teaching of mathematics courses. These ideas were also proposed by some of the teacher educators. These findings are broadly in line with the well-known recommendations by other mathematics educators (e.g., Ball et al., 2008).

Both the practicing teachers and the teacher educators argued that course contents are purely general knowledge for future teachers if there were no explicit links with school mathematics, and hence these contents were not regarded as useful for future teachers. The practicing teachers argued that the links between university and school mathematics were difficult to perceive if the university course contents were set at too high a level compared with the usual school contents. The findings show that the pure mathematical contents, i.e., the common content knowledge (CCK) of the mathematics teacher education, should be carefully examined so that the most relevant mathematical contents for future mathematics teachers could be discovered. It is well known that a weak knowledge of mathematics on the part of teachers has a negative influence on teaching (McDiarmid, Ball & Anderson, 1989), but, on the other hand, a competence solely in mathematics is insufficient enough for good teaching (Hodgen, 2011). It seems that the relevance of mathematics courses depends on how explicitly the link between university and school mathematics is stressed in mathematics courses.

The results suggest that, in addition to pure mathematical contents, mathematics teacher education should include mathematical contents designed specifically for teachers. The practicing teachers argued that they needed mathematical knowledge and skills that were different from the skills and knowledge useful for mathematicians. This reflects the well-known ideas embodied in pedagogical content knowledge (Shulman, 1987), or Specialized content knowledge, SCK (Ball et al., 2008), i.e., an area of knowledge for teachers that also separates researchers from teachers. The practicing teachers and teacher educators suggested that the mathematical courses should be at least partly separate for future teachers and mathematicians. In practice, this would mean that more resources would be needed for mathematics teacher education, which might be a challenge.

Many of the teacher educators who participated in this study espoused the traditional view of development that emphasizes improving future teachers’ subject matter knowledge (SMK) (Ball, 2003). Some educators viewed that good teaching requires knowledge of pure mathematics (CCK) and therefore they suggested that pure mathematical contents should be increased. On the other hand, many educators viewed that future mathematicians and future teachers need different kind of mathematical knowledge (SCK) and therefore they suggested that some of the present contents should be modified to be more suitable for teachers or new courses should be developed for teachers. Some educators viewed that forming integrated knowledge of pedagogy and mathematics (SCK) is one challenge for the mathematics teacher education, and therefore they suggested that the present courses should be re-scheduled or integrated.

On the other hand, a majority of the practicing teachers observed that there was a wider need for development than simply reforming the mathematical contents. The majority of practicing teachers demanded more courses concerned with teaching mathematics, students’ learning difficulties in mathematics, and how to differentiate mathematics teaching. These knowledge domains can be identified as Knowledge of content and teaching (KCT) and Knowledge of content and students (KCS). The practicing teachers pointed out that they needed to alternate the knowledge and skills of teaching and learning in many classroom situations, and they seemed to consider that
the KCS and KCT knowledge types were interconnected especially in classroom actions (see Fernández, Figueiras, Deulofeu, et al., 2011; Ball et al., 2008). Many practicing teachers considered that they had learned pedagogical and mathematical issues in the course of their teacher education and that they had found teaching practice a very useful experience, but still they had difficulty in forming an integrated understanding of pedagogy and mathematics (see also Korthagen & Kessels, 1999; Sharp, 2004).

This linkage of theory and practice (Carlson, 1999; Tryggvason, 2009) seems to be a major challenge in mathematics teacher education, since it concerns not only the pedagogical and mathematical studies but also the teaching practice. Earlier research work has shown that solving the problem will not be simple. According to Verloop, Driel, and Meijer (2001), it is still difficult to foresee how teacher knowledge can be clarified clear for future teachers in their teaching practice. One of the problems appears to arise from the teacher educators’ knowledge: not even experienced educators in the field of teaching practice have a clear grasp of the types of knowledge that teaching procedures involve, which makes it difficult to make the connection between theory and practice visible to student teachers (Verloop et al., 2001; Asikainen, Pehkonen & Hirvonen, 2013).

Filling the gap between theory and practice is a demanding task because teaching practice comprises only a small proportion of the teacher education studies as a whole. Hence, it is almost unrealistic to suggest that the gap could be fulfilled during the teaching practice. Our results suggest that the links between theory and practice should be made visible in all of the components of the teacher education so as to support future teacher development. Numerous suggestions have been made for the solution of this problem, e.g., by approaching it from practice to theory (Carlson, 1999), by developing the pedagogy of teacher education (Korthagen & Kessells, 1999), or by taking problem-solving into account in mathematics teacher education (Leikin & Levav-Waynberg, 2007). There is a possibility that contents, teaching methods, and the learning process may all be involved in the solution.

We have come to the realization that one of the key factors in reforming mathematical studies is the performance of a detailed analysis and comparison of curricula in university and school mathematics. In fact, it would seem obvious that the pure mathematical contents (CCK) should be the same as the topics in school mathematics or, at the very least, explicit links should exist between university and school mathematics. Another challenge is to design and develop special content knowledge (SCK) courses for future teachers. As yet, there is no general consensus about the knowledge and skills included in SCK (see e.g., Carrillo, Climent, Contreras & Muñoz-Catalán, 2013; Flores, Escudero & Carillo, 2013) but there should be no problem in designing new courses for future teachers, since there would be no harm caused if the subject matter and pedagogical contents are mixed. But as far as conceptualizing MKT is concerned, there is still work to be done to reach a consensus about this type of knowledge.

Although teacher education and teachers’ knowledge are related (Darling-Hammond, Chung, & Frelow, 2002), more research into the challenges revealed by individual teacher education programs will be needed in order to construct a broader picture of this multifaceted phenomenon. Individual reports may act as an important part in this process by evaluating and improving mathematics teacher education before all of the universal challenges facing mathematics teacher education have been fully recognized. Although the present study has concerned only a single mathematics teacher education program, we would suggest that the following issues may prove to be more general challenges facing all mathematics teacher education programs:

- The connections between university mathematics and school mathematics are not self-evident for student teachers. Teachers need pure mathematical knowledge, e.g. Common Content Knowledge (Ball et al., 2008) and Subject Matter Knowledge (Shulman, 1986;
1987). However, student teachers may find that the mathematics studied at university level is too advanced and has no clearly visible connections to the mathematics taught in school.

- Specific mathematical knowledge is missed from teacher education, while the contents of mathematical courses focus too largely on pure mathematics. In addition to mathematical content knowledge, teachers also need specific mathematical knowledge, e.g. Specialized Content Knowledge (Ball et al., 2008) or School Mathematics (O’Meara, 2010), because they need to carry out a variety of different activities (e.g., producing teaching materials, formulating and marking exams) for which pure mathematical knowledge is insufficient.

- Teachers may have too few tools to be able to teach “good and poor” students at the same time. In the classroom teachers are simultaneously attempting to evaluate their students’ starting levels, to recognize their individual learning habits, and also to implement different teaching strategies that will match up to the pertaining situation. The knowledge required in these situations can be referred to as Pedagogical Content Knowledge (Shulman, 1986; 1987) or as both Knowledge of Content and Students and Knowledge of Content and Teaching (Ball et al., 2008).

- Courses in teacher education may be too theoretical (e.g., Carlson, 1999; Korthagen & Kessells, 1999). Student teachers may feel that mathematical and pedagogical courses and also teaching practice are too far removed from teachers’ actual work.

The results of this study encourage us in the development work of mathematics teacher education although the circumstances are still difficult at the starting point. The most demanding part has been and will be to evaluate what the personnel in mathematics teacher education teach and what kind of methods they use. We believe that assessment, feedback, and the teacher education personnel themselves and their cooperation are important components in the process of improving teacher education. It is common sense that there is always a possibility of improvement, and therefore the development must begin from critical thinking: what can we do better? With this article, we should like to encourage other researchers to evaluate and develop teacher education, and hence we would close with words that are too frequently dead and buried:

- Without criticism, development dies –

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How Education Affects Mathematics Teachers' Knowledge: Unpacking Selected Aspects of Teacher Knowledge

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ABSTRACT
It is no surprise that all mathematics teacher education programs attempt to increase future teachers’ knowledge, since teachers’ knowledge has an effect not only on their teaching but also on their students’ achievements. However, measuring the relationship between teachers’ knowledge and their education is overly demanding. In this study we unpacked a selection of aspects of Mathematical Knowledge for Teaching (MKT) and created a 72-item survey that would measure teacher educators’ and graduated teachers’ perceptions of what graduated teachers have learned well or less well during teacher education. The data were collected under the auspices of the University of Eastern Finland. The results show that the teacher educators (N=18) and graduated teachers (N=101) who participated have developed rather similar perceptions of what graduated mathematics teachers learn well or poorly during their teacher education. According to the results, Subject matter knowledge and Pedagogical content knowledge receive similar emphases in the teacher education program, but certain fields of teacher knowledge, such as Common content knowledge, receive a greater emphasis than others. The approach described here provides a simple way of investigating this demanding phenomenon, and hence the limitations and the possibilities of the study will be discussed in detail.

Keywords: mathematical knowledge for teaching, MKT, teacher education, evaluation

INTRODUCTION
Evaluations are usually important for teacher education programs because they can reveal information about the development needs of mathematics teacher education (e.g. König, Blömeke, Paine, Schmidt, & Hsieh, 2011; Qian & Youngs, 2016). The contents of mathematics teacher education are related in various ways to future teachers’ knowledge, and future teachers’ knowledge is likewise related to students’ achievement in mathematics (Hill, Rowan, & Ball, 2005; Monk, 1994; Schmidt, Cogan, & Houang, 2011; Schmidt, Houang, & Cogan, 2011).
Hence, the contents or learning opportunities that a mathematics teacher education program can offer to student teachers are important topics in evaluations of the effectiveness of mathematics teacher education (Hsieh et al., 2011; Qian & Youngs, 2016). A survey can be a useful tool for assessing teacher education (Darling-Hammond, Chung, & Frelow, 2002; Schmidt, Houang, & Cogan, 2011). Surveys may, for example, capture teacher educators’ and student teachers’ perceptions of the kind of learning opportunities that teacher education has offered, which can be an important indicator for the contents of mathematics teacher education (Schmidt, Cogan, & Houang, 2011; Schmidt, Houang, & Cogan, 2011). The results of surveys investigating graduated teachers’ perceptions of what they think that they learned during their own teacher education can be an important indicator of teachers’ general preparedness and self-efficacy, i.e., their belief in their own ability to succeed in specific situations or accomplish a task. In addition, the results may help in the development of the contents of teacher education (Darling-Hammond, 2006). Thus, it can be claimed that teacher educators’ and graduated teachers’ perceptions of the contents or learning can be useful indicators of the development needs in the contents of teacher education in general (e.g. Darling-Hammond, 2006; Schmidt, Houang, & Cogan, 2011).

In light of this claim, to assess one of the teacher education programs used in Finland we created a 72-item survey that would measure teacher educators’ and graduated teachers’
views concerning what graduated teachers learn during their teacher education. The survey questions were directed at capturing respondents’ personal perceptions of graduated teachers’ learning, an approach that differs from measuring actual learning. There are some limitations inherent in self-reporting data, such as graduated teachers’ perceptions of their own learning may not reflect their actual learning. However, as Darling-Hammond’s (2006) study shows, teachers’ perceptions of their own learning can reflect their sense of self-efficacy and preparedness, both of which are related to teaching (Darling-Hammond, Chung, & Frelow, 2002; Hill et al., 2005). Hence it can be claimed that preparedness and self-efficacy are also important factors in teachers’ professional competence (Darling-Hammond, 2006).

In order to analyze in greater detail teacher educators’ and graduated teachers’ perceptions of graduated teachers’ learning, the survey was developed on the basis of a well-known teacher knowledge framework known as Mathematical Knowledge for Teaching (MKT). In this case, “greater detail” refers to the investigation of teacher knowledge within the framework of six subcategories, as developed in the MKT. Teacher educators’ and graduated teachers’ viewpoints together provide an insight into the nature and extent of the Subject matter knowledge and Pedagogical content knowledge that graduated teachers have gained during their teacher education. The first three research questions were the following:

1. On the basis of their course contents, what mathematical knowledge for teaching do teacher educators believe that their student teachers learn?
2. After completing all of the courses in their mathematics teacher education, what mathematical knowledge for teaching do graduated teachers believe that they have learned?
3. From previous appointed viewpoints, in which ways do teacher educators’ and graduated teachers’ perceptions converge?

The responses to the first two questions define the nature of the Subject matter knowledge and Pedagogical content knowledge that are emphasized to a greater or lesser extent in the teacher education program under evaluation. The third research question attempts to locate similarities between teacher educators’ and graduated teachers’ views. When, for example, the teacher educators consider that the graduated teachers have learned something well or less well, it cannot necessarily be claimed that the graduated teachers also think that they have learned about the same topic in the same way.

Curricula in mathematics teacher education seem to have an impact on teachers’ knowledge (Schmidt, Houang, & Cogan, 2011; Qian & Youngs, 2016). In the present study, the mathematics majors have taken twice as many mathematics courses as have mathematics minors. Since the number and also the content of the mathematics courses will have differed, this may well have an impact on graduated teachers’ views concerning what they have learned well or less well. Hence, the responses to the fourth research question will indicate whether different curricula in fact affect graduated teacher perceptions of their own learning.
4. How do mathematics majors and minors consider their learning of different topics in the course of their education?

We would claim that the results of this kind of evaluation can be implemented in at least two ways to improve the teacher education program that has been studied. Firstly, knowing how graduated teachers have learned about different fields of teacher knowledge provides an opportunity for us to improve weak parts and to maintain the strong parts of teacher education. Secondly, recognition of the potential differences between mathematics majors’ and minors’ perceptions of their learning will provide us with an opportunity to develop our future curricula in teacher education.

INTERNATIONAL ASSESSMENT OF MATHEMATICS TEACHER EDUCATION

Teacher Education and Development Study in Mathematics (TEDS-M) was the first international empirical study using large samples to explore the differences between teacher education programs in seventeen countries (Bankov et al., 2013; Brese & Tatto, 2012; Ingvarson et al., 2013; Tatto et al., 2008). Approximately 5,000 teacher educators and 22,000 future teachers from 750 programs in some 500 teacher education institutions in 17 countries participated in the TEDS-M study. Beliefs and views held by teacher educators’ and future teachers’ were explored to form a coherent picture of the characteristics of teacher education programs with respect to policy, schooling, and social contexts at the national level. Perhaps the most interesting part of the TEDS-M study was the exploration of the differences between future teachers’ Mathematics content knowledge and Mathematics pedagogical content knowledge in TEDS-M countries, since in the meantime a lot of research has been published explaining these results (e.g., Blömeke, Hsieh et al., 2011; König et al., 2011; Suhl & Kaiser, 2011; Schmidt, Houang, & Cogan, 2011; Qian & Youngs, 2016).

Based on TEDS-M, a variety of similarities and differences can be found in all teacher education programs. The teacher education programs included three types of courses connected to 1) mathematical knowledge, 2) pedagogical knowledge related to the teaching of mathematics, and 3) general pedagogical knowledge more generally related to instructional practices and schooling (Schmidt, Cogan, & Houang, 2011). However, according to findings published by Schmidt, Cogan, and Houang (2011), these three areas are emphasized differently in the different TEDS-M countries. According to the TEDS-M reports (Ingvarson et al., 2013; Tatto, et al., 2012), a period of teaching practice was included in most programs, but fewer included field experience of learning about issues involving school organization and management. In general, programs that focused on preparing teachers to become higher level teachers (lower and upper secondary level) provided opportunities for learning contents in greater depth than teachers preparing for the primary level.

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1 The countries participating in the TEDS-M study included Botswana, Canada, Chile, Chinese Taipei, Georgia, Germany, Malaysia, Norway, Oman, the Philippines, Poland, Russia, Singapore, Spain, Switzerland, Thailand, and the USA.
The results of the TEDS-M suggest that future teachers’ knowledge and the content of teacher education are related. Teacher education programs will offer different opportunities for learning that refers to the design of curricula. Qian and Youngs (2016) found that when future teachers acquire mathematical and pedagogical knowledge there is also a relationship between these opportunities to be learned. The content of taken mathematics courses has a more powerful effect on future teachers’ Mathematics content knowledge than the number of the courses mastered (Qian & Youngs, 2016). The curriculum in teacher education seems to have an effect on future mathematical and pedagogical knowledge, yet there were differences between the various TEDS-M countries\(^2\), which suggests that for some countries the development of curricula might have even more effect.

The combination of the TIMSS and TEDS-M results reveals that teachers’ knowledge and students’ achievement are linked. The results describing 8th grade achievements in TIMSS-2003 and the Teacher knowledge scores in TEDS-M-2010 were transformed onto the same scale\(^3\) (Schmidt, Houang, & Cogan, 2011). A strong correlation ($R^2=0.70$, $P<.0004$) between student achievement and teacher knowledge was found, suggesting that when future teachers’ Mathematics content knowledge was high, the country was more likely to succeed in the TIMSS. The future teachers’ Mathematics content knowledge and Mathematics pedagogical

\(^2\) Statistically significant differences were not the same for all countries.

\(^3\) These results can be compared since the theoretical frameworks of the TEDS-M and TIMSS were partly similar.
content knowledge and the students’ achievement rating are shown in Figure 1. Figure 1 has been constructed from data provided by Schmidt, Houang, and Cogan (2011), to which we added the TEDS-M scores related to Mathematics pedagogical content knowledge (Center for Research in Mathematics and Science Education, 2010).

The ratio of the courses related to Mathematics, Mathematics pedagogy, and General pedagogy seem to play a highly significant role in teacher education. Schmidt, Houang, and Cogan (2011) discovered that when future teachers’ knowledge as measured by their TEDS-M scores was either higher or lower than, or alternatively in line with, students’ achievement in TIMSS, then the detectable differences were in the ratio of the courses in three areas⁴. Those countries who succeed better in the TEDS-M than in the TIMSS (shown as marked “M+” in Figure 1), emphasized especially Mathematics in their curricula (49%/31%/21%). The countries that succeeded better in the TIMSS than the TEDS-M (shown as “M−” in Figure 1) emphasized more General pedagogy but less Mathematics (37%/36%/28%). The countries with most success in both the TEDS-M and the TIMSS (shown as “M” in Figure 1) emphasize was between others (42%/33%/26%). How these three areas are stressed in teacher education seems to bear a relationship to teacher knowledge and student achievement. The results indicate that finding the balance of course-work in these three areas may be the key issue in any attempt to improve existing teacher education programs.

König et al. (2011) introduced two alternative models for interpreting the results of the TEDS-M. In their study they compared future teachers’ General pedagogical knowledge scores in the TEDS-M in the contexts of Germany, Taiwan, and the USA. The USA had lower scores in Mathematics pedagogical content knowledge than Germany and Taiwan (see Figure 1), and the same was true for future teachers’ scores in General pedagogical knowledge. König et al. (2011) found that the USA, when compared with Taiwan and Germany, had weaker scores in items that were measuring recalling and understanding, but better scores in items measuring generating. The second model shows that, when compared with Taiwan and Germany, the USA had similar scores for items of assessment, weaker scores for items of adaptivity, but better scores for items of structure and management. These sub-areas, whose General pedagogical knowledge was constructed in the TEDS-M, describe in greater details the areas in which the USA would probably benefit from development. In other words, Figure 1 is unable to fully describe the required development, but if the Mathematics content knowledge and Mathematics pedagogical content knowledge are split up into their more detailed components, we may nevertheless discover ways in which these respective teacher education programs can be improved. The division of the results into smaller components may also be necessary before the results of the TEDS-M can be employed in the improvement of teacher education:

⁴ Three areas are connected with 1) mathematical knowledge, 2) pedagogical knowledge related to the teaching of mathematics, and 3) general pedagogical knowledge related to instructional practices and schooling more generally.

⁵ Numbers referred to the proportions of Mathematics, Mathematics pedagogy, and General pedagogy.
This also suggests that this kind of division might be a key factor in other development programs to find the parts need to be improved.

**MATHEMATICAL KNOWLEDGE FOR TEACHING**

Many models of teacher knowledge (e.g., Ball, Thames, & Phelps, 2008; Ernest, 1989; Fennema & Franke, 1992; O’Meara, 2010; Rowland, Turner, Thwaites, & Huckstep, 2009) reveal the kind of knowledge needed for teaching mathematics, while some of them also suggest ways in which Subject matter knowledge and Pedagogical content knowledge can be divided. *Mathematical Knowledge for Teaching (MKT)* divides teacher knowledge into six components, where three components constitute Subject matter knowledge and three components represent Pedagogical content knowledge. Although the origins of MKT are American, in more recent times the MKT methods of measurement have also been used beyond the American context, for instance in Ireland (Delaney, Ball, Hill, Schilling, & Zopf, 2008), South Korea (Kwon, Thames, & Pang, 2012), Ghana (Cole, 2012), Indonesia (Ng, 2012), Norway (Fauskanger, Jakobsen, Mosvold, & Bjuland, 2012), Iceland (Jóhannsdóttir & Gísladóttir, 2014), and Malawi (Kazima, Jakobsen, & Kasoka, 2016). Interest has also been growing in applying this framework within the Nordic context (Fauskanger et al., 2012; Jankvist, Mosvold, Fauskanger, & Jakobsen, 2015; Jóhannsdóttir & Gísladóttir, 2014; Mosvold, Bjuland, Fauskanger, & Jakobsen, 2011; Mosvold & Fauskanger, 2013).

Within this framework, Subject matter knowledge and Pedagogical content knowledge are divided into three more detailed domains, see Figure 2. Subject matter knowledge does not require Pedagogical content knowledge, whereas Pedagogical content knowledge requires

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**Figure 2.** Domains of Mathematical Knowledge for Teaching (MKT) by Ball et al. (2008)
Subject matter knowledge (Ball et al., 2008). There are at least two ways to read this. Firstly, Marton and Booth (1997) consider that mathematics is simply mathematics, but we cannot simply teach it because we always teach something. Secondly, if we think about how to organize coursework for future teachers in teacher education, teachers first need at least some understanding of mathematics before they can fully understand Pedagogical content knowledge.

**Subject matter knowledge**

In many mathematics teacher education programs, student teachers have mathematics courses that are intended to serve both future teachers and also mathematicians. The contents of these courses are usually “pure mathematics”, i.e., concerning knowledge of concepts, results, and proofs in different areas of mathematics. In terms of MKT, these aspects of Subject matter knowledge are referred to as Common content knowledge (CCK) for teachers. Hence, CCK can be seen as mathematical knowledge that is not unique to teaching and hence it is also useful in other professions (Ball et al., 2008; Hill, Ball, & Schilling, 2008).

According to Ball et al. (2008), teachers need mathematical knowledge that is unique to teaching, for example, evaluating tasks, designing mathematical problems, and marking exams. Hence, this knowledge has been termed Specialized content knowledge (SCK) for teachers. Hill et al. (2008, pp. 377-378) describe SCK as a competence that “allows teachers to engage in particular teaching tasks, including how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems.” In mathematics teacher education, this could mean that student teachers are prepared, for instance, to formulate, mark, and grade exams. Marking exams is, for example, a specific teaching task but it does not require Pedagogical content knowledge and hence marking exams requires Specialized content knowledge.

Mathematics teachers need to master the structure of mathematics in terms of how concepts are related, how concepts form different topics, and how the structure of mathematics is constructed in relation to the topics. In addition, teachers need to know how concepts can be presented to students at different levels at school. For example, the concept of function might be presented to elementary students in the form of an idea. From one school level to another, definitions then become more exact and develop toward more formal versions. This knowledge of mathematical structures and awareness of how mathematical topics are related to each other is known as Horizon content knowledge (HCK) for teachers (Ball & Bass, 2009; Ball et al., 2008).

**Pedagogical content knowledge**

If we understand how people learn, we can understand how to teach better. The idea behind Pedagogical content knowledge in MKT is similar, because knowledge of teaching and of learning are separated. Knowledge of content and students (KCS) is more like knowledge of how students learn mathematics: A teacher must be able to anticipate students’ difficulties and
misconceptions, to hear and respond to students’ thinking, and to choose suitable examples while teaching (Ball et al., 2008). Furthermore, a teacher might be better able to motivate and inspire students if s/he knows what students are interested in. Hill et al. (2008) consider that KCS is a primary element in Shulman’s (1986) PCK, because one part of Shulman’s PCK (1986, p. 9) is “an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons”. Thus, KCS is an amalgam of knowledge of students, learning, and mathematics.

According to Ball et al. (2008) teachers choose effective strategies, arrange a classroom, and make decisions when they plan their teaching, which all require Knowledge of content and teaching (KCT). Furthermore, if students are asking for the right answer to a given problem, a teacher can hide the answer and try to provide an opportunity for them to discover the answer for themselves. Decisions of this kind require pedagogical thinking; which answers can reasonably be given to students directly and which should reasonably be left hidden for later discovery. Sometimes there are opportunities in a classroom that allow students to make mathematical discoveries by themselves, and thus a teacher needs to decide if he/she is going to depart from the original plan or not. All of these aspects are connected to KCT. In general, KCT is required in both planning and teaching, and it is definitely something that teacher education should prepare students for.

The content of curricula and the requirements of teachers’ work are mostly interconnected and hence knowing these aspects is important for teachers. Knowing the content of curricula is referred to as part of Knowledge of Contents and Curriculum (KCC) for teachers. A wider perception of KCC also includes knowledge of teaching materials (such as textbooks, other materials, etc.), teaching instruments (blackboards, overhead projectors, etc.), and technology (computers, smart boards, calculators, software, etc.). The use of technology in teaching requires, for instance, amalgam knowledge of mathematics, pedagogy, and technology (Koehler & Mishra, 2009). These aspects of KCC are also something for which teacher education is trying to prepare future teachers.

METHOD

Context

The University of Eastern Finland offers two programs for students of mathematics: one for mathematicians and another for teachers. Both programs are almost identical with respect to mathematics courses, but they differ in terms of minor subjects. In the following, we describe the Mathematics teacher education program.

The Mathematics teacher education program includes the Bachelor’s degree (180 cp) and Master’s degree (120 cp). Both degrees are required in Finland for the qualification of a

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6 One credit point (cp) is equivalent to 25 hours of study. The recommendation is to take 60 cp per year.
mathematics teacher. The program includes mathematical studies (120 cp), pedagogical studies (60 cp), and studies in one or two minor subjects (60 cp each). Most mathematical studies are traditional mathematics courses relevant to both teachers and mathematicians (e.g., calculus, analysis, algebra, differential equations). The pedagogical studies include theoretical studies of teaching and learning (30 cp), the didactics of mathematics (10 cp), and teaching practice (20 cp). Student teachers can study any school subjects as a minor subject, but their typical choices are physics or chemistry, or both. In its entirety, the program provides a qualification to teach mathematics and minor subjects in lower or upper secondary schools and vocational schools.

It is notable that teachers who have studied mathematics as major or minor are regarded as possessing the same competence to teach mathematics at secondary school level, even though their studies are to some extent different. The major difference is in the number of mathematical courses. The number of courses is doubled for future teachers who have studied mathematics as a major (mathematics majors) in comparison to future teachers who have studied mathematics as a minor (mathematics minors) since mathematics majors take advanced-level mathematical studies (60 cp) in addition to their basic and intermediate mathematical studies (60 cp).

Instrument

Based on the literature, we produced 72 statements about knowledge related to teaching mathematics (e.g., Ball, 2003; Ball et al., 2008; Ball & Forzani, 2009; Ball & Hill, 2008; Hill et al., 2005; Hill, Schilling, & Ball, 2004; Sleep, 2009). However, we faced some challenges in unpacking the conceptualization of MKT theory in practice. For instance, answering students’ Why?-questions is listed as one type of the mathematical tasks confronted in teaching, which is explicitly classified as Specialized content knowledge (Ball et al., 2008, p. 400). However, we would also suggest that the nature of the knowledge that teachers require in such situations depends on the kind of Why?-questions that students ask. Some of the students’ questions are not related to mathematics at all. If a student asks, for example, “Why is this theorem true?” a teacher will probably answer by using Common content knowledge. If the teacher’s answer does not satisfy the student, s/he will ask supplemental Why?-questions that will enable them to “find examples to make a specific mathematical point”, or to “link representations to underlying ideas and to other representations” (Ball et al., 2008, p. 400). In that case, the answer will require Specialized content knowledge as presented by Ball et al. (2008). In addition, an understanding of why some topics are central to a discipline and others are not (Ball et al., 2008, p. 391) or an understanding of the structure of mathematics requires an explanation of why some mathematical topics are important for gaining the total picture. Hence, in our opinion, Horizon content knowledge or Knowledge of content and curriculum may also be involved in the search for answers to students’ Why?-questions. At present, the domain definitions of MKT are not unambiguous, and hence more attention should be paid to clarifying them and make them more accessible. Relevant examples of this knowledge, its purpose, and its meanings are needed in order to close the gap in the current literature.
In consequence, we put our effort into unpacking the MKT domains as statements about the knowledge required for teaching mathematics. The statements were designed so that half were connected with Subject matter knowledge and half with Pedagogical content knowledge in terms of MKT. The themes and categorization were not presented in the survey, and the placement of items was mixed. It will be pertinent at this stage to describe how these 72 items involving the themes in various ways are related to the kinds of mathematical knowledge that may be needed for teaching mathematics. The numbers in brackets refer to the number of items concerned with a current theme.

### Table 1. Themes and items concerned with Common content knowledge

<table>
<thead>
<tr>
<th>Themes</th>
<th>Content of items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Calculus</td>
<td>Differential calculus</td>
</tr>
<tr>
<td></td>
<td>Integral calculus</td>
</tr>
<tr>
<td>Functions</td>
<td>Functions and equations</td>
</tr>
<tr>
<td></td>
<td>Polynomial functions</td>
</tr>
<tr>
<td></td>
<td>Root and logarithmic functions</td>
</tr>
<tr>
<td></td>
<td>Trigonometric functions</td>
</tr>
<tr>
<td>Geometry</td>
<td>Analytic geometry</td>
</tr>
<tr>
<td></td>
<td>Geometry</td>
</tr>
<tr>
<td>Data and probability</td>
<td>Probability theory</td>
</tr>
<tr>
<td></td>
<td>Statistics</td>
</tr>
<tr>
<td>Numbers and vectors</td>
<td>Number sequences</td>
</tr>
<tr>
<td></td>
<td>Vector calculus</td>
</tr>
<tr>
<td>Mathematical concepts and notations</td>
<td>Exact use of mathematical notations</td>
</tr>
<tr>
<td></td>
<td>Mathematical concepts</td>
</tr>
<tr>
<td>Studying skills</td>
<td>Solving mathematical exercises and problems</td>
</tr>
<tr>
<td></td>
<td>Reading and understanding university-level course materials used in mathematical studies</td>
</tr>
<tr>
<td>Mathematical methods</td>
<td>Mathematical calculation methods</td>
</tr>
<tr>
<td></td>
<td>Mathematical reasoning rules (e.g., logic)</td>
</tr>
<tr>
<td></td>
<td>Use of suitable mathematical methods</td>
</tr>
<tr>
<td></td>
<td>Use of figures, diagrams, and models to promote own mathematical thinking</td>
</tr>
</tbody>
</table>

Common content knowledge (see Table 1). A teacher requires, more than anything else, a knowledge of a range of different mathematical topics. In Finland, for example, the mathematical topics are prescribed in the national curriculum for each grade (e.g., The Finnish National Board of Education, 2003). Twelve statements outline the mathematical topics, which cover roughly all of the necessary topics at secondary school level in the country (The Finnish National Board of Education, 2003). The topics are concerned with calculus (2), functions (4), geometry (2), data and probability (2), and numbers and vectors (2). Because Common content knowledge is actually more extensive than a knowledge of the prescribed topics, some of the items are also connected with mathematical concepts, notations (2), and methods (4). Two of the statements are also connected with study skills, since both student teachers and teachers
need to possess the requisite skills for solving mathematical problems and also the ability to
read mathematical texts (2).

Specialized content knowledge (see Table 2). In the classroom, teachers present mathematics
in a variety of ways, and this process requires a knowledge of mathematical representations
and strategies useful in visualizing mathematics (3). Teachers also need to have skills that
enable them to formulate new assignments and exams, and to evaluate and grade their
students’ output (3). Sometimes a teacher in the classroom will attempt to justify the learning
of mathematics, and hence it may well be advantageous for the teacher if he or she knows
something about the history and philosophy of mathematics (2) or how mathematics has
influenced everyday life, culture, and society (2).

Horizon content knowledge (see Table 3). In very general terms, mathematics can be said
to be concerned with concepts and with the relationships between them. In consequence, it
might be said that mathematics is ultimately a single accurate construction made up of
mathematical concepts. In this light it is clear that a teacher needs to possess knowledge of
mathematical structures (3). However, experts and novices may well see the structure of

<table>
<thead>
<tr>
<th>Themes</th>
<th>Content of items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Presenting mathematics</td>
<td>Mathematical representations (verbal, picture, symbolic)</td>
</tr>
<tr>
<td></td>
<td>Use of multiple representations (verbal, picture, symbolic) of the same mathematical</td>
</tr>
<tr>
<td></td>
<td>entity</td>
</tr>
<tr>
<td></td>
<td>Use of visualizations in mathematics</td>
</tr>
<tr>
<td>Formulation and marking assignments</td>
<td>Formulation of mathematical exams</td>
</tr>
<tr>
<td></td>
<td>Formulation of mathematical exercises</td>
</tr>
<tr>
<td></td>
<td>Marking exam responses</td>
</tr>
<tr>
<td></td>
<td>Recognition of straightforward and more problematic mathematical exercises</td>
</tr>
<tr>
<td>Influence of mathematics</td>
<td>Mathematical applications in everyday life, science, and technology</td>
</tr>
<tr>
<td></td>
<td>Impact of mathematical development on culture and society</td>
</tr>
<tr>
<td>History and philosophy of</td>
<td>History of mathematics</td>
</tr>
<tr>
<td>mathematics</td>
<td>Philosophy of mathematics</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Themes</th>
<th>Content of items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematical structures</td>
<td>Hierarchy of mathematical concepts (e.g. axioms, definitions, lemmas, propositions)</td>
</tr>
<tr>
<td></td>
<td>Fields of mathematics in general</td>
</tr>
<tr>
<td></td>
<td>Structure of concepts in different mathematical fields (e.g. how mathematical</td>
</tr>
<tr>
<td></td>
<td>concept of limit is related to other concepts)</td>
</tr>
<tr>
<td>Learning of mathematical structures</td>
<td>Teaching the structure of mathematics</td>
</tr>
<tr>
<td></td>
<td>Recognition of previous and forthcoming mathematical concepts in teaching</td>
</tr>
<tr>
<td></td>
<td>mathematics</td>
</tr>
<tr>
<td></td>
<td>Students’ actual mathematical know-how at different school levels (e.g., typical</td>
</tr>
<tr>
<td></td>
<td>issues in the need for revision with students)</td>
</tr>
</tbody>
</table>
mathematics differently, and therefore teachers need to know how the structure of mathematics can be seen differently before teachers can intervene to support the understanding of mathematical structures (3).

Knowledge of content and teaching (see Table 4). A teacher needs knowledge and skills concerned with the planning of his/her teaching (2). To become effective in his/her teaching, a teacher should probably take into account how students generally learn mathematics. Thus, teachers’ professional skills may also develop if they apply their knowledge of learning theories to their own teaching (2). In the classroom teachers need access to various skills connected, for example, with discussing mathematics (3) and using examples (2). To achieve the aims of their teaching, teachers may also need to underline the most significant issues and explain the aims of the teaching (3). In general, an effective teacher will endeavor to generate a positive experience of the general process of learning mathematics (2).

Knowledge of Contents and Students (see Table 5). Teachers need to evaluate students’ behavior, and their thinking and speaking at the level of the individual (5). Teachers may be able to identify students’ misconceptions of mathematics and also prevent them, if they possess some knowledge of the challenges posed by the learning of mathematics (4). Teachers need to know about supporting and motivating students in their learning (2), because ultimately the teachers are attempting to improve students’ mathematical competence (3).

Knowledge of content and curriculum (see Table 6). In most countries, curricula are regarded as defining specific learning objectives, and hence curricular knowledge will also provide relevant knowledge for a teacher (2). Technology, textbooks, and other equipment will be useful tools for both learning and teaching. Teachers will require this variety of knowledge when selecting and using textbooks in their teaching of mathematics (2). The same
will be true of knowledge and skills concerned with the use of technology in teaching mathematics (2).

Two electronic surveys were produced. The survey given to the graduated teachers was intended for mathematics teachers who had already graduated from the Mathematics teacher education program. The question presented was in the form “What rating would you give to the knowledge / skills that you learned from your teacher education program?” The teachers evaluated the 72 presented knowledge or skills items on a five-point Likert scale (1 = not at all, 2 = poor, 3 = fair, 4 = good, 5 = excellent). The respondents were asked to avoid evaluating any

7 Because of the Finnish language, there was need to distinguish between the aspects of knowledge and skills issues, since the term knowledge usually refers to “information” in the Finnish language. Hence, half of the statements referred to knowledge and half to skills. The order of the items was selected so that the respondents would first evaluate teacher knowledge issues and then issues concerned with teacher skills.

<table>
<thead>
<tr>
<th>Themes</th>
<th>Content of items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Improving students’ competence</td>
<td>Developing students’ problem-solving skills</td>
</tr>
<tr>
<td></td>
<td>Developing students’ mathematical thinking</td>
</tr>
<tr>
<td>Evaluating students’ competence</td>
<td>Evaluation of students’ competence</td>
</tr>
<tr>
<td></td>
<td>Students’ problem-solving and mathematical reasoning skills demonstrating their deep understanding</td>
</tr>
<tr>
<td></td>
<td>Evaluation of the accuracy and coherence of students’ conclusions</td>
</tr>
<tr>
<td></td>
<td>Recognition of students’ meaningful talk in their learning of mathematics</td>
</tr>
<tr>
<td></td>
<td>Assessment of students’ mathematical knowledge, skills, and ability for their future studies</td>
</tr>
<tr>
<td>Challenges of learning</td>
<td>Recognition of students’ errors and taking them into account in teaching</td>
</tr>
<tr>
<td></td>
<td>Prevention of students’ misconceptions</td>
</tr>
<tr>
<td></td>
<td>Recognition of students’ misconceptions in mathematics (e.g. multiplication increases and division decreases results)</td>
</tr>
<tr>
<td></td>
<td>Recognition of learning difficulties in mathematics</td>
</tr>
<tr>
<td>Motivation and support skills of learning</td>
<td>Motivating students to learn, understand, and know mathematics</td>
</tr>
<tr>
<td></td>
<td>Supporting mixed-level students in learning mathematics</td>
</tr>
<tr>
<td></td>
<td>Supporting students’ self-confidence in their mathematical skills</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Themes</th>
<th>Content of items</th>
</tr>
</thead>
<tbody>
<tr>
<td>Use of textbooks</td>
<td>Use of textbooks in teaching mathematics</td>
</tr>
<tr>
<td></td>
<td>Choosing a suitable textbook or textbook series suited to one’s own teaching style</td>
</tr>
<tr>
<td>Curricular knowledge</td>
<td>Objectives and contents of the national mathematical curriculum</td>
</tr>
<tr>
<td></td>
<td>Evaluating one’s own teaching with respect to the national curriculum</td>
</tr>
<tr>
<td>Use of technology</td>
<td>Use of technology in teaching mathematics</td>
</tr>
<tr>
<td></td>
<td>Using technology in one’s own teaching of mathematics</td>
</tr>
</tbody>
</table>
knowledge and skills that they may have gathered since graduation or in the course of their teaching.

The survey for teacher educators was aimed at all educators currently working in the Mathematics teacher education program. The question was in the form “Based on your course contents, what rating would you give to the knowledge / skills that your students have learned?” The educators were asked to evaluate the 72 knowledge or skill issues on the same five-point Likert scale as that used with the graduated teachers (1 = not at all, 2 = poor, 3 = fair, 4 = good, 5 = excellent). The respondents were also asked to take into account all of the courses that they taught.

According to Ihantola and Kihn (2011) pretesting, clear standard instructions, avoiding abstract concepts, the order of questions, and the length of survey should be considered in the design of survey. Therefore, first four independent researchers improved wordings for instructions and seventy-two statements. The order of questions and the length of survey were carefully considered before the survey was piloted. Two persons from the study group and two persons out of the study group piloted the survey. Because the persons out of study group are usually better in evaluating the intelligibility of wordings, and the abstractness of used concepts, they were used as well. All the eight participants gave their opinion about the intelligibility of instructions and statements, and proposed suggestions for developing the survey during pretesting and piloting. Based on pre-testing, only few words were replaced because of their manifold meanings.

Sample

The data were collected in the course of 2012–2013. The survey intended for the graduated teachers was sent to all 187 mathematics teachers who had graduated from the University of Eastern Finland in the period 2002–2012, and our sample (N=101) eventually included 54% of them. Seventy-two per cent (73) of the respondents in the sample had taken mathematics as their major and 28% (28) as their minor. Apart from one respondent, all of them had experience of teaching mathematics at school level or they were currently working as teachers. The survey intended for the teacher educators was sent to all 24 educators in the field of mathematical studies, pedagogical studies, and teaching practice. Our sample (N=18) comprises 75% of all those who were working in the Mathematics teacher education program in 2012. Seventy-eight per cent (14) of the teacher educators were currently teaching mathematics (Math-Educators), while and 22% (4) of them were working in pedagogical studies or teaching practice (PT-Educators).

Analysis

The data were analyzed as follows. Four mean values for each statement were calculated. The first mean value was calculated from the survey data acquired from the graduated teachers. The mean stands for the perceptions of all graduated teachers of how they had personally learned each of the issues mentioned in the survey. In addition, three mean values were
calculated for the survey data provided by the teacher educators. These were the perceptions of all teacher educations, the perceptions of the Math-Educators, and the perceptions of the PT educators of how their students had learned each of the issues. Because the actual learning aims were usually different for the mathematics and PT educators, their views are presented separately even though there were only a handful of educators involved in pedagogical studies and teaching practice.

The statistical difference between those teachers who had studied mathematics as their major or as their minor was explored. A difference was expected because the quantity of mathematical studies for the major-subject student teachers is double that taken by the minor-subject student teachers. The Mann-Whitney U-Test was used to explore the statistical differences between the groups (e.g., Sheskin, 2003).

Cronbach Alpha test is widely used for testing the consistency of an instrument. Higher Cronbach Alpha scores denote stronger correlation between tested items (Cronbach, 1951; Tavakol & Dennick, 2011). Since surveyed items were designed in the way that the statements were related to the different MKT domains, the consistency of statements were tested within these categories. The number of graduated teachers’ survey responses was fivefold compared to the teacher educators’ survey responses, therefore graduated teachers’ responses were used for counting the Cronbach’s Alpha scores. The Cronbach’s Alpha score for the items dealing with Subject matter knowledge was .931, while for those dealing with Pedagogical content knowledge it was .950 (for further details, see Table 7). These values predict that the items correlated highly within a single component and hence the instrument may be regarded as reliable. In addition, Alpha score .931 in subject matter area gives .13 error variance (random error) and Alpha score .95 in Pedagogical content knowledge gives .10 error variance (see Tavakol & Dennick, 2011).

The dependencies between the perceptions of the teacher educators and the graduated teachers were explored. Their correlation was calculated solely on the basis of the mean values. In another words, the correlation between two mean values was tested for all 72 items. First, a normal distribution of data was tested using Komogorov-Smirnov and Shapiro-Wilk. The correlation between the mean values for the teacher educators and graduated teachers surveys

**Table 7.** Cronbach Alpha scores for the teacher knowledge components in the survey

<table>
<thead>
<tr>
<th>Number of items</th>
<th>Subject matter knowledge</th>
<th>Cronbach’s Alpha (α)</th>
</tr>
</thead>
<tbody>
<tr>
<td>37</td>
<td>Common content knowledge</td>
<td>.931</td>
</tr>
<tr>
<td>20</td>
<td>Specialized content knowledge</td>
<td>.736</td>
</tr>
<tr>
<td>11</td>
<td>Horizon content knowledge</td>
<td>.781</td>
</tr>
<tr>
<td>35</td>
<td>Pedagogical content knowledge</td>
<td>.950</td>
</tr>
<tr>
<td>15</td>
<td>Knowledge of content and teaching</td>
<td>.885</td>
</tr>
<tr>
<td>14</td>
<td>Knowledge of content and students</td>
<td>.921</td>
</tr>
<tr>
<td>6</td>
<td>Knowledge of content and curriculum</td>
<td>.668</td>
</tr>
</tbody>
</table>

Table 7. Cronbach Alpha scores for the teacher knowledge components in the survey
was then calculated. In the present case, since the data were normally distributed, use was made of Pearson’s correlation (Hauke & Kossowski, 2011).

RESULTS

We now present in two parts the results of the survey for the graduated mathematics teachers and the teacher educators involved in the Mathematics teacher education program. Firstly, the results of how the graduated teachers and teacher educators see that graduated teachers have learned different issues during teacher education will be presented. To assist readers to acquire a more structured picture of the results the items detailed in the surveys were categorized by means of the framework supplied by Mathematical Knowledge for Teaching and also on the basis of their different themes. Secondly, because the number and content of mathematical courses were different for those teachers who had taken mathematics as either their major or minor, respectively, analytical attention will be paid to their statistical difference.

How have the graduated teachers learned different issues during their teacher education according to the perceptions of the teacher educators and the graduated teachers themselves?

Perceptions of Common content knowledge

Teacher educators’ and graduated teachers’ responses to items of Common content knowledge is represented in two parts – mathematical topics (Figure 3) and a number of
general issues, such as using representations, methods, and concepts (Figure 4). Apart from theme, data, and probability, the items surrounding themes are close together when the graduated teachers’ perceptions opt for a sequence that ranges from the highest to the lowest scores (see Figure 3). The graduated teachers considered that they possessed a better knowledge of calculus than they did of numbers and vectors; at the same time, they thought that they possessed a better knowledge of functions than they did of geometry. In terms of theme, data, and probability two items remain remote from any others, suggesting that the graduated teachers considered that they had learned probability theory better than they had statistics.

The views held by the graduated teachers and PT-Educators about learning the topics of mathematics are generally quite similar. In contrast, Math-Educators’ perceptions are similar to the graduated teachers’ perceptions but differed noticeably for the themes of geometry and of data and probability.

The topics of geometry and probability theory are covered in the Mathematics teacher education program by special courses. This means that there are only a small number of Math-Educators who teach these topics. Because fewer Math-Educators teach these topics, this may have some impact on the average values of the Math-Educators. In contrast, no special course in statistics is included in the teacher education program, which we can see from the results in the form of a low means. Both the graduated teachers and the teacher educators were of the
opinion that the graduated teachers had not learned statistics as effectively as they had other topics. Since statistics is a compulsory topic for teachers at school level, it would be reasonable to argue that teachers also need to know statistics. Hence, any improvements of this particular teacher education program should include the establishment of a special course in.

Figure 4 shows that according to the respondents’ perceptions, the graduated teachers have acquired a better knowledge of mathematical concepts and notations than they have of mathematical methods. PT-Educators’ and Math-Educators’ views intersect in the items labeled Reading and understanding the university-level course materials of mathematical studies and Using mathematical notations correctly. Study skills involving Reading and understanding course materials should be regarded as a prerequisite for learning mathematics at university, but teaching practice may not be the best time for learning this skill. Thus, it seems to be reasonable that the Math- and PT-Educators rate this item more highly than do the PT-Educators. Recognizing the accurate use of mathematical notations may well be easier for PT-Educators, since Math-Educators generally deliver lectures, whereas PT-Educators work more closely with student teachers.

Perceptions of Specialized content knowledge

Items on the themes almost overlap when the chosen sequence concerns the graduated teachers’ perceptions ranked from the highest to the lowest scores (Figure 5). The graduated teachers considered that they had a better knowledge concerned with presenting mathematics than with how mathematics has influenced everyday life, culture, and society. In addition, they also suggested that they had a better knowledge of formulating and marking assignments than they did of the history and philosophy of mathematics.

The PT-Educators’ opinions concerning the graduated teachers’ learning in items related to SCK were generally more positive than those expressed by the Math-Educators or graduated teachers. In addition, the perceptions of all of the teacher educators were similar to those of all of the graduated teachers.

The graduated teachers considered that their learning of the items History of mathematics, Philosophy of mathematics, and Impact of mathematical development for culture and society had been modest during teacher education. The themes of these items are connected with general mathematical knowledge, which can be useful for teachers, but they may not be a prerequisite for actually teaching mathematics. Within the theme of Formulation and marking assignments, the graduated teachers considered that they possessed rather similar skills in formulating mathematical exercises, identifying easy and difficult mathematical exercises, and formulating and marking exams. These skills are connected with the measuring aspect of mathematical competence, which may in fact be more vital knowledge than that connected with those previously considered. The theme of presenting mathematics received the highest scores based on graduated teachers’ perceptions. The mathematical representations and skills involved in visualizing mathematics are clearly vital in the teaching of mathematics.
When comparing the graduated teachers’ perceptions of how they had learned issues related to Common content knowledge and Specialized content knowledge, it may be noted that the mean values are generally considerably lower for specialized content knowledge than for Common content knowledge. In other words, the graduated teachers considered that they had learned pure mathematical issues better that were also useful for other professions, but at the same time they had acquired only a modest mathematical knowledge that was specifically related to teaching mathematics.

**Perceptions of Horizon content knowledge**

The graduated teachers had experienced more effective learning related to the theme concerning mathematical structures than they had concerning the Learning of mathematical structures. The opinions of the Math-Educators and PT-Educators intersect in the item identified as Fields of mathematics in general (Figure 6). Interestingly, the items on the left, to which the Math-Educators had assigned higher values, represent a theoretical knowledge of mathematics, while the items on the right, to which the PT-Educators assigned higher values, are concerned with practical knowledge. These observations seem to be in line with the intention of those particular studies, since mathematical studies do in fact usually have theoretically-oriented aims, while teaching practice has practically oriented aims.
Perceptions of Knowledge of content and teaching

The themes seem to be mixed in the domain of Knowledge of content and teaching, where the chosen sequence consists of the graduated teachers’ perceptions ranked from the highest scores to the lowest (see Figure 7). In consequence, the themes will be analyzed one by one.

The difference between the views expressed by the Math- and PT-Educators is clear in the domain of Knowledge of content and teaching (Figure 7). The PT-Educators’ views are more positive than those of the Math-Educators. Since mathematical studies are connected principally with Subject matter knowledge, while pedagogical studies are connected primarily with Pedagogical content knowledge, the results can be regarded as in line with the intentions of these studies.

Analyzing themes one by one in their theme planning, the graduated teachers considered that they had acquired better skills related to planning single lessons than they had with regard to planning complete courses. This finding is logical, since within the Mathematics teacher education program the student teachers planned individual classes during their teaching practice, but they devoted less time to planning complete courses. The PT-Educators produced high values in relation to the learning of these items, while the values produced by the Math-Educators were low. This is also understandable since PT-Educators teach student teachers to plan lessons whereas Math-Educators do not.
In the context of the theme concerned with Learning theories, the graduated teachers thought that they had a better knowledge of constructivism and its application to their own teaching than they achieved when applying other learning theories. Because constructivism can be seen as the basis of modern learning theory, the result is positive. Nevertheless, research is continuously proceeding further, and new research-based results concerned with learning can be of benefit to teachers. The graduated teachers considered that they possessed the lowest skills when they attempted to apply information gained from research-based results to their teaching. Hence, increasing the contents related to the skills of applying knowledge gleaned from research-based results concerned with learning mathematics to real-life teaching may well be a perspective that could help in the development of teacher education.

According to the graduated teachers, the mean values for items drawn from other themes, such as explaining the aims of teaching, discussing mathematics, and using examples,
are quite similar to each other. The items in these themes are more like inward-oriented action than providing teachers with theoretical knowledge. In addition, producing reports of positive experiences received in the process of learning mathematics relates to working in classrooms. Nevertheless, the scores in this domain are almost the lowest obtained.

Perceptions of Knowledge of content and students

One clearly defined theme—improving students’ competence—emerged, whereas other themes in Knowledge of content and students are more mixed (Figure 8). In consequence, the themes will be analyzed one by one.

The difference between the views held by the PT- and Math-Educators in this domain is clear. The PT-Educators’ views are more positive than those of the Math-Educators. Since mathematical studies are generally connected with Subject matter knowledge, while pedagogical studies are concerned primarily with Pedagogical content knowledge, the results are in line with the aims of these studies. However, the mean values for all of the teacher
educators are noticeably higher than those of all of the graduated teachers. The educators seem to have a more positive view of the learning of the graduated teachers than those of the graduated teachers themselves.

Analyzing themes one by one, the graduated teachers considered that they possessed better skills for improving students’ competence than they did for other themes. Certainly, ways of developing students’ mathematical thinking and problem-solving skills are undoubtedly useful knowledge for mathematics teachers.

In the theme concerned with evaluating students’ competence, the graduated teachers considered that they possessed more useful skills for assessing students’ competence than they did skills concerned with evaluating students’ reasoning, conclusions, or meaningful talk. It is possible that the evaluation of students’ competence can be understood generally as simply grading students, but evaluating students’ competence on the basis of the students’ reasoning, conclusions or meaningful talk may be a more accurate undertaking that can be regarded as in-action knowledge, since that type of evaluation usually occurs in the classroom.

In relation to the theme Challenges of learning the graduated teachers considered that they possessed better skills with regard to recognizing students’ mistakes than recognizing students’ misconceptions or learning difficulties. On some occasions, recognizing mistakes requires simply Common content knowledge, but a knowledge of misconceptions and learning difficulties in mathematics requires more than simply pure mathematical knowledge (Ball et al., 2008). The results indicate that graduated teachers may have better skills appropriate for recognizing a wrong answer, but only moderate skills with regard to students’ specific challenges in learning.

In relation to the theme of motivation and supporting skills, the graduated teachers considered that they possessed better skills for motivating students to learn mathematics than they did for handling mixed-level students’ learning. Indeed, motivation skills are important for a teacher, but in the classroom there are always students with good or poor mathematical skills, and hence it is obviously important that the teacher should know how to handle mixed-level students.

With respect to this component of teacher knowledge, the mean values are under three for every item within Knowledge of content and students according to the graduated teachers’ observations. This particular aspect of the study is clearly lower in comparison with other components in the area of MKT. In developing mathematics teacher education, all of the issues connected with Knowledge of Content Students may well be worth considering.

**Perceptions of Knowledge of content and curriculum**

The differences between the PT- and Math-Educators’ views are the highest in this domain (Figure 9). Curricular knowledge and the use of textbooks seem to increase the difference between them. Math-Educators normally use their own teaching materials rather than textbooks, and hence it is reasonable that their views are lower. Issues related to national
Within the theme of the use of textbooks, the graduated teachers considered that they possessed better skills related to using a textbook in their teaching, compared with the requisite skills involved in choosing a textbook best suited to their teaching style. The usual textbooks are used in teaching practice, but choosing the right textbook is also an important skill, since in Finland teachers have the right to choose the textbooks/series that they wish to use. Thus it is reasonable to argue that this decision requires knowledge about the differences between the different textbooks. Choosing the right textbook is the first thing that a graduated teacher needs to do after graduation. Hence, exploring the differences between textbooks or studying textbooks with an eye on teaching at school may well be something to consider when developing the teacher education program.

Within the theme of technology, the graduated teachers apparently consider that their knowledge of technology is at approximately the same level as their skills related to using technology in their teaching. Knowledge of national curricula and of the skills involved in evaluating teaching within the context of the national curriculum have similar mean values. Technology and its use in teaching is growing strongly nowadays at school level and hence it would be worth considering whether this is sufficient.

**Overall results: Differences between the components of teacher knowledge**

According to the perceptions of the graduated teachers, the average of the mean values in the field of Subject matter knowledge is 3.24, while that of the mean values in the field of
Pedagogical content knowledge is 2.67. These results indicate that the graduated teachers thought that they had learned about the various issues in Subject matter knowledge slightly better than in Pedagogical content knowledge. However, the more accurate components of MKT reveal that the graduated teachers felt they had learned the strongest issues in the context of Common content knowledge. The mean values for each component in MKT are presented in Figure 10. The next best values are related to issues in Knowledge of content and teaching. Almost the same average values pertain for the domains of Horizon content knowledge and Knowledge of content and curriculum. Graduated teachers evaluated they had learned the modest issues in Specialized content knowledge and in Knowledge of content and students.

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The PT-Educators’ perceptions are generally more positive than those expressed by the Math-Educators (Figure 10), but if we take into account the number of teacher educators in these two areas, it can be seen that the mean value for all of the teacher educators seems to be more in line with the perceptions of the graduated teachers (Figure 11). The Pearson correlation test indicates similar conclusion. For all 72 of the items, there was significant positive linear correlation ($r=.502$, $p<.000007$). This result means that both the teacher educators and the graduated teachers have quite similar perceptions of what graduated teachers have learned well and what they have learned less well in the course of their teacher education.

Comparisons between the domains of MKT reveal that the role played by the different types of knowledge is not included in this teacher education program. Rather, some of the MKT components are stressed more than others, while there remain some components that
would deserve greater emphasis. Thus, if we wish to change how these fields of teacher knowledge are emphasized in the future, it will be important to know how these components of teacher knowledge are currently handled.

Ways in which mathematics majors or minors thought about what they had learned about different issues during their education

The Mathematics majors had taken twice as many mathematics courses during their teacher education as had the mathematics minors. Because the number and contents of the mathematics courses were significantly different for the mathematics majors and minors, the statistical differences were explored with the aid of the Mann-Whitney U-test. The results show that the mathematics majors considered that they had learned the following issues better than the mathematics minors (Table 8).

The mathematics majors felt that they had learned thirteen items better than had the mathematics minors. One of them was related to Pedagogical content knowledge and twelve to Subject matter knowledge. Most of the items fell into Common content knowledge.

The mathematical courses labeled Geometry, Number sequences and History of mathematics are optional in the studied teacher education program, but in our experience, most mathematics majors take them. These single special courses seem to have had an effect on the graduated teachers’ views. The differences between the contents of the mathematics courses that the mathematics majors and minors had taken explain the observed differences. However,
by reading course materials and using mathematical methods or notations, a knowledge of the structure of mathematics may well prove to be knowledge and skills components that will improve if the individual student takes more courses involving university mathematics. Hence, it can be claimed that the number of mathematics courses could explain these differences.

Interestingly, apart from one item, all of the differences occurred in Subject matter knowledge, and especially in the domain of Common content knowledge. In addition, no differences occurred in the context of Knowledge of content and students or Knowledge of content and teaching. Thus, results of this kind help to provide a picture of the kind of courses that their teacher education has offered these teachers. In other words, the results indicate the impact of the different curricula on mathematics majors and minors: a wider mathematics curriculum seems to have a greater impact on the graduated teachers’ views concerning their learning Subject matter knowledge and less concerning their learning Pedagogical content knowledge.

DISCUSSION

In this study we have explored graduated mathematics teachers’ perceptions of how they have learned different issues during their teacher education, and also teacher educators’ perceptions of how they view the ways in which graduated teachers have learned the same
issues during their teacher education. In order to examine these components of teacher knowledge more precisely, the present survey was developed by using the teacher knowledge framework widely known as Mathematical Knowledge for Teaching (MKT). Differences can be found in the smaller fields of teacher knowledge (König et al., 2011), and therefore breaking Subject matter knowledge and Pedagogical content knowledge down into their smaller components can reveal some unexpected issues.

Based on the results, the teacher educators offering pedagogical courses generally have a more optimistic perception of how the graduated teachers have learned different issues during their teacher education compared with the perceptions expressed by the teacher educators of mathematical courses. However, the views of all of the teacher educators and graduated teachers are quite similar at a general level (see 5.1.7). In consequence, all of the teacher educators and graduated teachers can be referred to using the term both groups.

These two perceptions of the two groups have informed our results concerning the ways in which Subject matter knowledge and Pedagogical content knowledge are emphasized in one particular teacher education program in Finland. The results indicate that in this teacher education program Subject matter knowledge and Pedagogical content knowledge are emphasized almost equally. However, when we divided teacher knowledge into its more detailed components it was interestingly revealed that smaller fields of teacher knowledge are stressed differently.

**Strengths and challenges in Subject matter knowledge**

Based on the results, Common content knowledge is clearly the most strongly emphasized component of Subject matter knowledge in the studied teacher education program. According to the perceptions of both groups, the graduated teachers had learned pure mathematical issues better however, there were some exceptions, such as statistics, which does not belong to the present teacher education program.

Our results also indicate that issues related to Specialized content knowledge seem to be emphasized less than those concerned with Common content knowledge in the present teacher education program. Both groups considered that the graduated teachers had generally learned issues in Common content knowledge better than they had issues concerned with Specialized content knowledge. In the context of Specialized content knowledge, both groups felt that the graduated teachers had learned issues concerned with presenting mathematics, formulating and marking assignments more proficiently than issues related to the history and philosophy of mathematics or issues dealing with the ways in which mathematics has influenced everyday life and society.

Common content knowledge is undoubtedly an area of fundamental knowledge for teachers, and it is a fact that without knowledge of this kind teaching mathematics is impossible. A study published by Copur-Gencturk and Lubienski (2013) indicates that mathematics content courses have an impact on future teachers’ Common content knowledge,
but at the same time courses of this kind may not actually make the Subject matter knowledge the most relevant for effective teaching. Their results indicate, rather, that Specialized content knowledge is also needed for effective teaching. Copur-Gencturk and Lubienski (2013, p. 219) claim rather convincingly that “perhaps specialized content knowledge should actually be considered a subset of common knowledge if it simply adds a teaching-specific story line around the everyday mathematics content.”

The narrative context of mathematics might be presented, for example, if a teacher knows something about the history of mathematics and how mathematics has influenced the development of culture, society, or everyday life. Our results indicate that it is precisely this area of knowledge that remains less emphasized than knowing about mathematics content. In order to improve the present teacher education program, one procedure might be to lay greater stress on mathematical contents that take into account the special characteristics of teachers’ work and can be referred to under the heading of Specialized content knowledge. Perhaps designing a new hybrid course where the fields of teacher knowledge are mixed would be a possible option for developing this aspect of the present teacher education program. Hybrid courses seem to have a good effect on future teachers’ Common content knowledge and Specialized content knowledge, whereas the content courses only impact on Common content knowledge (Copur-Gencturk & Lubienski, 2013).

Strong aspects and challenges in Pedagogical content knowledge

Based on our results, Knowledge of content and teaching is the component of Pedagogical content knowledge that is most strongly highlighted in the present teacher education program. As the results show, items scored differently within the themes in Knowledge of content and teaching, which suggests that such items should be investigated, theme by theme. In Knowledge of content and teaching, both groups thought that the graduated teachers had, for example, learned more about planning single lessons than about planning complete courses. Similarly, they appeared to have acquired a better knowledge of constructivism than of other learning theories, and about presenting questions than answering them. These results are not dramatic, since planning a course usually includes planning several lessons. Constructivism might also be considered more important than other learning theories, and presenting questions is generally considerably easier than answering them, since presenting can be planned in advance, but answering cannot. Despite this, the overall results indicate that Knowledge of content and students has received less emphasis than Knowledge of content and teaching. In Knowledge of content and students, both groups thought that the graduated teachers possessed a better knowledge concerning the improvement of students’ problem-solving skills and mathematical thinking but weaker knowledge concerned with recognizing students’ misconceptions or learning difficulties.

Teachers do, of course, sometimes recognize which mathematical exercises are difficult for students, but they do not consistently know why these exercises are difficult for their students (Hill et al., 2008). However, if a teacher does know why some students are
experiencing difficulties in specific exercises; it might be possible to approach these students differently. Indeed, teachers require knowledge related to both teaching and learning, since these knowledge types are closely connected in many classroom situations. Observation of students’ learning can be an effective starting point for changing one’s teaching style, or knowing how students learn mathematics, or understanding the kinds of challenges that they might face at the individual level may very well provide some of the keys for teaching more effectively. In the present state of teacher education, knowledge concerning the ways in which students learn mathematics seems to less emphasized than knowledge about teaching mathematics per se, and hence increasing Knowledge of content and students may well be worth considering.

The impact of a wider mathematics curriculum

Based on our results, the mathematics majors considered that they had learned some issues better during their teacher education than had the mathematics minors (see 5.2). Statistical differences occurred in thirteen of the seventy-two items. These differences appeared, for example, in knowledge of geometry, number sequence, and the history of mathematics. Generally speaking, the mathematics majors had completed courses in the studied teacher education program referred to as Geometry, Number sequence, and the History of mathematics, whereas the minors had rarely done so, which may help to explain these statistical differences.

Qian and Youngs (2016) found that the contents of mathematical courses have an effect in TEDS-M countries on future teachers’ knowledge. Their results indicate that the content of mathematics content courses impacts on future teachers’ Mathematics content knowledge. However, the number of mathematics courses taken also has an effect on future teachers’ Mathematics content knowledge. Their study indicates that it is important not only how many courses a teacher education program offers but also what kind of contents these courses concentrate on. If teachers have studied a wider or different mathematics curriculum in their teacher education, this may also have an effect on their Mathematics content knowledge.

Based on our results, it was interesting to note that all except one of the differences occurred in Subject matter knowledge, particularly in Common content knowledge. The curricular difference between mathematics majors and minors may be a contributing cause of these differences. A wider curriculum for mathematics majors seems to have had a greater effect on how graduated teachers felt that they had learned Subject matter knowledge but less effect on their sense of what they had learned about Pedagogical content knowledge. We consider that these results help to form a picture of the kind of course contents that mathematics majors have acquired more than mathematics minors. Because the curriculum can have an effect on future teachers’ knowledge (Qian & Youngs, 2016), there is opportunity for improving teacher education by reforming the curriculum for future teachers.
Reassessing the study and context

External validity refers to the generalizability and transferability of research findings. It is especially concerned with generalizing results by applying them to other samples, time periods, and settings (Ihantola & Kihn, 2011). In the present study, the teacher educator and practicing teacher sample sizes were comprehensive and the samples were representative. All of the respondents originated from a single teacher education program and hence these findings primarily describe the actual circumstances of the program that was investigated. In the present study the mean values of the graduated teachers’ perceptions suggest that there are topics that are learned well and less well. Interestingly, the mean values related to the teacher educators’ perceptions indicate the same conclusion. The positive correlation between the teacher educators’ and graduated teachers’ perceptions suggests that the two distinct study groups evaluated the graduated teachers’ learning in similar ways, resulting in a strong significance ($r=.502$, $p<.001$). Since the two distinct study groups felt that the graduated teachers may have learned some topics differently than others, our actual findings indicate that at least some potential breakdowns must have occurred in the teaching and learning. However, further research is needed to discover how suitable the present course contents actually are for future teachers. Earlier studies have shown that the perceptions of teachers and their students regarding their teaching and learning may exist in a complex set of relations. Furthermore, these perceptions may also be reflected in the teaching outcomes and students’ learning (see Trigwell, Prosser, & Waterhouse, 1999).

The internal validity of the study is closely connected with the logic linking research and existing theory and with valid conclusions drawn on the basis of the study (Ihantola & Kihn, 2011). In our opinion, MKT provides a highly useful teacher knowledge framework for evaluating mathematics teacher education, but more research attention needs to be directed toward clarifying the descriptions of the six MKT domains. Missing still is a clear sense of which items are likely to provide good representations for each domain. In this vein, Ball et al. (2008, p. 403) have pointed out that “How to capture the common and specialized aspects of teacher thinking, as well as how different categories of knowledge come into play in the course of teaching, needs to be addressed more effectively in this work.” Greater attention also needs to be paid to the ways in which different kinds of knowledge can be accurately classified in terms of the six MKT categories requires as well. We tend to agree with Markworth, Goodwin, and Glisson (2009) that there are pros and cons when MKT is applied in the investigation of the principal characteristics of teacher knowledge. In order to evaluate the kind of knowledge learned by student teachers during their teaching practicum course, Markworth et al. (2009) used the domains of MKT to encode their interview responses and conversational topics. By using MKT in their analysis, they were able to encapsulate information in rather greater detail related to the kind of knowledge that the student teachers had gained during the course. Markworth et al. (2009, p.70) have suggested, however, that MKT “categories are static and do not reflect their use in practice and that the boundaries between categories are sometimes fuzzy”. This is the so-called boundary problem of MKT, which has also been noted by Ball et
al. (2008): "Related to this is a boundary problem: It is not always easy to discern where one of our categories divides from the next, and this affects the precision (or lack thereof) of our definitions. We define common content knowledge as the mathematical knowledge known in common with others who know and use mathematics, but we do not find that this term always communicates well what we mean." (p.403). Thus, if the theory does not communicate effectively to the research community, researchers may perceive the various conceptualizations rather differently. According to Hoover, Mosvold, Ball, and Lai (2016), this has indeed occurred in a number of studies.

In our opinion, MKT provides a highly useful teacher knowledge framework for evaluating teacher education programs in a Finnish context, since similar contents can be recognized in the Finnish teacher education coursework compared with the components of MKT. In Finnish teacher education, purely mathematical courses are usually intended for both future teachers and mathematicians. On the other hand, pedagogical mathematics courses are intended primarily for future teachers, and the content of these courses takes the special characteristics of teachers' work more into account. This division is similar to that in MKT between Common content knowledge and Specialized content knowledge, where the first part is useful for any profession requiring mathematics, while the second part is specialized, making it more appropriate for mathematics teachers (Ball et al., 2008).

In Finnish teacher education some of the contents of pedagogical courses focus on mathematics teaching, while others focus more on the process of learning mathematics. Thus, for example, some contents focus on learning mathematics, i.e., learning difficulties, and learning theories, while other contents focus on teaching mathematics in terms of such aspects as the different teaching methods. All of these contents in Finnish teacher education can be seen to possess a similar point of view in terms of Knowledge of content and students and Knowledge of content and teaching in MKT. In addition, curricular knowledge and also the use of equipment in teaching mathematics are usually concentrated in pedagogical courses, while the mathematical contents related to the structure of mathematics are covered in mathematics courses. All of the components of MKT contain viewpoints that are similar to the contents of mathematics teacher education in Finland. Based on these perspectives, we think that MKT fits into the Finnish context well, especially as the main background theory for evaluating mathematics teacher education.

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Using network analysis methods to investigate how future teachers conceptualize the links between the domains of teacher knowledge

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Teacher knowledge frameworks rarely describe how domains of knowledge are related to each other in the minds of teachers. In the present study, future teachers perceptions about “the kind of knowledge required for teaching mathematics” were classified with the Mathematical Knowledge for Teaching (MKT) framework and relationships between raised issues were examined with the aids of network analysis. According to the results, the six MKT domains exist in hierarchical order in the minds of future teachers. The findings indicate that broadening the research on studying how knowledge is related in the minds of teachers may profit in a novel way.

1 Introduction

Subject teachers need to possess strong content knowledge since they have to be able to discuss the topic, answer students’ questions, and offer a variety of explanations (Shulman, 1986). Robust subject knowledge alone does not make a teacher a good teacher; teachers also need knowledge in order to be able to transform the relevant subject knowledge for their students. According to Shulman (1986), this process involves Pedagogical Content Knowledge that stretches beyond subject knowledge. In the 1980s, Pedagogical Content Knowledge was a paradigm that was missing: there was no interest in the process of the knowledge transformation that teachers implement when teaching (Shulman, 1986). Nowadays, however, it is clear that Shulman’s (1986, 1987) ideas and conceptualizations have been used as a starting-point in research into teacher knowledge. Shulman presented two questions in his first paper that he considered to be salient in the scrutiny of teacher knowledge.
What are the domains and categories of content knowledge in the minds of teachers? How, for example, are content knowledge and general pedagogical knowledge related? (Shulman, 1986, p. 9)

Questions such as these have guided several researchers in the field of mathematics education research (e.g. Ball, Thames, & Phelps, 2008; Ernest, 1989; Fennema & Franke, 1992; O’Meara, 2011; Rowland, Turner, Thwaites, & Huckstep, 2009). Although these researchers have presented varying frameworks for teacher knowledge in the context of mathematics, the frameworks nevertheless have a number of features in common. The researchers agree that teachers need strong subject knowledge, in this case in the form of a knowledge of mathematics. They all share the view that mathematics teachers also need a different kind of knowledge from that required by mathematicians. Furthermore, they all consider that pedagogical knowledge is needed in the teaching of mathematics. By presenting the categories or domains of teacher knowledge, they all also answered Shulman’s first question about the nature of the domains and categories involved in teacher knowledge. Hence, to answer Shulman’s second question, there is a need to consider how the domains and categories of teacher knowledge are related to each other.

O’Meara (2011) takes a solid stand on the second question. In his Ladder of Knowledge framework, the first step is subject matter knowledge, the second is pedagogical knowledge, and the last one is a knowledge of effective teaching. O’Meara (2011) develops a strong argument in response to the question of how such knowledge is related: “...it does not make good sense... to make teachers believe that they can make a full scale assault on pedagogical content knowledge without first acquiring a strong content knowledge” (O’Meara, 2011, p. 192).
Similarly, Rowland et al. (2009) respond to the second question in their Knowledge Quartet. In the Knowledge Quartet, mathematical knowledge and beliefs refer to Foundation Knowledge, while each of the other three domains is based on this. Familiarity with the relevant Foundation Knowledge is a premise for teachers intending to teach mathematics. Rowland et al. (2009) explain that such knowledge is in its very nature different from the other domains because it refers to the knowledge that people hold whether they use it relevantly or not.

According to Mathematical Knowledge for Teaching (MKT), teachers require six different kinds of knowledge in their teaching of mathematics (Ball, et al., 2008). The MKT framework is based on Shulman’s conceptualization (1986), and its foundations are based in the American context. Nowadays, however, the use of MKT as a framework for teacher knowledge has been largely distributed to other countries as well, e.g. South Korea (Kwon, Thames, & Pang, 2012), Ghana (Cole, 2012), Indonesia (Ng, 2012), Norway (Fauskanger, Jakobsen, Mosvold, & Bjuland, 2012), Finland (Koponen, Asikainen, Viholainen, & Hirvonen, 2016), and Malawi (Kazima, Jacobsen, & Kasoka, 2016). Morris, Hiebert, and Spitze (2009) propose that MKT offers the most encouraging solution to the question of the different types of content knowledge needed for teaching mathematics well.

However, one issue in MKT has yet to be solved. Ball, Thames and Phelps (2008) note that many of the demands posed by teaching mathematics require a knowledge of the intersection of six knowledge types in MKT. Ball, Thames and Phelps (2008) illustrate this problem with an example: In other words, recognizing a wrong answer is common content knowledge (CCK), whereas sizing up the nature of an error, especially an unfamiliar error, typically requires nimbleness in thinking about numbers, attention to patterns, and flexible thinking about meaning in ways that are distinctive of specialized content knowledge (SCK). In contrast, familiarity with common errors and deciding which of several errors students are most likely to make are examples of knowledge of content and students (KCS). (p. 410). On the basis of
this perspective, it can be claimed that the domains of MKT are interrelated in a variety of ways. However, how the domains of MKT are related in teachers’ minds has yet to be studied.

From a position located thirty years later, we have posed Shulman’s (1986) two questions again but we have now organized our study to find answers related to the teacher knowledge framework of MKT. Thus, in the present study the research questions are the following:

1. According to views expressed by future teachers, what kind of knowledge is needed for teaching mathematics?

2. How are knowledge domains related to each other in the minds of future teachers?

To answer these two research questions, we asked student teachers (N=18) to write an essay about the kind of knowledge needed for teaching mathematics. Investigating this particular group proved interesting, since the student teachers had finished almost all of their studies and hence the impact of their teacher education ought to be visible in their expressed views. By using theory-driven analysis, the contents of the eighteen essays were converted into a single large network, and the patterns and structure of the network were then examined with the aid of Gephi software. By using such network analysis methods, we were able to detect a number of novel and unexpected structural features of teacher knowledge.

2 Theoretical framework

Mathematical Knowledge for Teaching (MKT) is a practice-based theory of mathematics teacher knowledge (e.g. Ball & Bass, 2009; Ball et al., 2008; Ball & Forzani, 2009; Ball, Hill, & Bass, 2005; Hill, Ball, & Schilling, 2008; Hill, Rowan, & Ball, 2005; Hill, Schilling, & Ball, 2004; Sleep, 2009). According to MKT, teachers need six different kinds of knowledge for teaching mathematics (Figure 1).
Common Content Knowledge (CCK). Teachers need a broad, competent knowledge of mathematics, e.g. mathematical theories, concepts, terms, definitions, rules, and symbols. Furthermore, they should be able to derive concepts, calculate, and prove theorems. All of these aspects of subject knowledge are important for teachers, as they would also be for other related occupations such as that of mathematicians. Hence, these aspects are termed Common Content Knowledge (Ball et al., 2008; Hill et al., 2008;).

Specialized Content Knowledge (SCK). Teachers commonly use a variety of mathematical representations, as well as visualizing mathematics and creating connections between the ways in which mathematics can be represented. If teachers have a historical knowledge of mathematics or if they know how mathematics can be applied, they will be able to use this information in their teaching (Jankvist, Mosvold, Fauskanger, & Jakobsen, 2015; O’Meara, 2011). Furthermore, teachers need to select relevant examples and appropriate exercises for every situation and also to evaluate how the examples and exercises work in practice. All of these aspects require mathematical knowledge of a kind that is unique to teaching. Hence, such aspects are generally termed Specialized Content Knowledge (Ball et al. 2008).
Horizon Content Knowledge (HCK). Teachers require knowledge of the structure of mathematics, such as how concepts are hierarchically related and how concepts together form topics. On the other hand, teachers also need to be aware of how mathematics is constructed for their students, e.g. which concepts students already know and which concepts they will learn later. In other words, teachers need to know which kind of prior knowledge will be needed for the student to learn new areas of mathematics. These aspects are related both to the structure of mathematics and also to the structure of learning mathematics, and hence they are regarded as aspects of Horizon Content Knowledge (Ball & Bass, 2009; Ball et al., 2008).

Knowledge of Content and Students (KCS). Because teachers need to gain an understanding of how their students learn mathematics in theory, they must also have some knowledge of learning theories. On the other hand, teachers need to recognize if students are liable to face challenges in learning mathematics or if they face particular challenges in their learning, such as possessing learning difficulties. Teachers need to know their students, understand their approaches to learning, and also recognize the various kinds of challenges that they face. In sum, teachers need to know some of the ways in which they can motivate their students and promote their learning. These aspects require a knowledge of how students think, know, or learn particular content. This aspect of teacher knowledge is referred to as Knowledge of Content and Students (Ball et al., 2008; Hill et al., 2008)

Knowledge of Content and Teaching (KCT). Teachers need to know and choose teaching methods for each situation. Hence, teachers need to possess knowledge concerning planning lessons, communicating, and promoting interaction in the classroom. Furthermore, a teacher needs to be able to change his/her teaching strategy, organize special learning support for students, and improve his/her own teaching. All of these aspects are related to planning and organizing the kind of teaching that requires an amalgam of knowledge of teaching and of
mathematics, and hence this knowledge is referred as Knowledge of Content and Teaching (Ball et al., 2008).

Knowledge of Content and Curriculum (KCC). The national curriculum normally supplies guidelines for specific teaching, and hence teachers should also be familiar with the contents of their national curriculum. In addition, however, teachers should have knowledge and skills relevant to the use of teaching materials (such as textbooks, other materials, etc.), teaching instruments (blackboards, overhead projectors, etc.), and technology (computers, smart boards, calculators, software, etc.). All of these aspects of knowledge can be summed up in terms of Knowledge of Content and Curriculum (Ball et al., 2008; Jankvist et al., 2015).

3 Method

3.1 Context and data collection

The University of Eastern Finland offers two different kinds of programs for future mathematics teachers. The subject teacher program provides the conventional qualification of a mathematics subject teacher. The second program is for so-called hybrid mathematics teachers. It provides a dual qualification for prospective mathematics subject teachers and also primary school teachers. Hence, students in the hybrid mathematics teacher program take more pedagogical studies than those aiming to become conventional mathematics teachers. Both of these teacher education programs offer masters-level degrees that are required for recognized teachers to work in Finland.

The data for this study was collected during a pilot course called Analysis skills for teaching mathematics, aimed at both future subject teachers and also future hybrid teachers. The new idea included in this pilot course was that the course instructor first taught graph theory for two hours and then the student teachers were required to analyze his/her teaching during the next two hours. This cycle was repeated eight times during the pilot course.
At the outset of the pilot course the student teachers wrote essays under the rubric of *The kind of knowledge needed for teaching mathematics*. Then student teachers were informed that one aim for the pilot course is to learn to reflect their own ideas related to the knowledge needed for teaching mathematics. Therefore, essays were returned to the student teachers and they were asked to share their thoughts and discuss the topic of essays in small groups. Furthermore, the student teachers were also requested to make notes about these conversations, and the course instructor encouraged the student teachers to “steal” others’ ideas but only if stolen idea matches their own thinking. These two-hour small group sessions were repeated altogether three times during the pilot course, and each time with mixed small groups which means that every student teacher had an opportunity to discuss everybody at least once. At the end of course, the student teachers were asked to reflect their ideas, this time individually, by writing essays about the kind of knowledge needed for teaching. It is, in consequence, very possible that the student teachers learned something new about teacher knowledge during the pilot course that they took, but in each situation the course instructor highlighted that student teachers must reflect their own thinking and approve all new ideas. Furthermore, the contents of the pilot course were related to the topic of graph theory and analysis of the teaching of graph theory, and throughout the course the teacher did not refer to teacher knowledge itself. In the present study, we look in depth at the final essays that the student teachers wrote.

A total of 18 student teachers participated in the study. Of these participants, 8 were future subject teachers and 10 future hybrid teachers (see Table 1). At the stage at which they participated in the course, they had completed almost all of the compulsory mathematical studies and over half of their pedagogical studies and teaching practice. Pedagogical studies focus on teaching and learning in general but also on the didactics of mathematics. Teaching practice is performed under guidance of experts (i.e. often PhDs in mathematics education) and it is undertaken in actual classrooms with real students. All participants had gathered teaching
experiences and in average, they were in their final year of study during the data collection. Some of student teachers in both groups had also already some experience of working in schools. Since both of the student teacher groups were at a relatively similar stage in their studies, in this paper we pay no attention to any of the small differences between the two groups.

Table 1: Background information about the student teacher participants (N=18).

<table>
<thead>
<tr>
<th></th>
<th>N</th>
<th>Stage of studies in years</th>
<th>Completed mathematical studies</th>
<th>Completed pedagogical studies</th>
<th>Completed teaching practice</th>
<th>Complete studies altogether</th>
<th>Do you have experience of working as a school teacher?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Future subject teachers</td>
<td>8</td>
<td>90% (4.5 of 5 years)</td>
<td>85% (103 of 120 cp)</td>
<td>60% (36 of 60 cp)</td>
<td>67% (2.75 of 4 stages)</td>
<td>78% (234 of 300 cp)</td>
<td>Yes* for 38% (3/8)</td>
</tr>
<tr>
<td>Future hybrid teachers</td>
<td>10</td>
<td>80% (4.0 of 5 years)</td>
<td>78% (93 of 120 cp)</td>
<td>64% (77 of 120 cp)</td>
<td>60% (2.4 of 4 stages)</td>
<td>84% (252 of 300 cp)</td>
<td>Yes* for 50% (5/10)</td>
</tr>
</tbody>
</table>

*Refers to temporary posts and total work experience ranging between a few hours and a month.

3.2 Analysis

After the student teachers’ essays had been read, *Conventional Content Analysis* was used to sort their individual issues into data-based categories (Hsieh & Shannon, 2005). This meant that if a particular issue was repeated in the essays, for example, the student teachers mentioned that teachers needed to “know mathematical theories”, it was classified as a single data-based issue. Next, the data-based issues were classified based on the various domains of MKT. This so-called theory-based classification is known as *Direct Content Analysis* (Hsieh & Shannon, 2005). Because both *Conventional* and *Direct Content Analysis* were applied, this part of the analysis is theory-driven in its entirety. The latter part of the analysis is basically similar to that reported in Markworth, Goodwin and Glisson (2009) study, where the interview responses and conversational topics were coded on the basis of the domains of MKT. In addition to
classification, the study involved the tracking of how the issues were related to each other (examples are shown in Table 2). The analyses yielded a total of 136 knowledge issues and 364 relations.

**Table 2: The knowledge issues required for teaching mathematics were classified into the various domains of MKT and the contents of the essays were converted into networks to enable later network analysis.**

<table>
<thead>
<tr>
<th>Sample section of essay</th>
<th>Data-based classification (Conventional Content Analysis)</th>
<th>Theory-based classification (Direct Content analysis)</th>
<th>Network</th>
</tr>
</thead>
<tbody>
<tr>
<td>A teacher should make mathematics meaningful. A teacher should connect the contents he/she is trying to teach to everyday life and also provide more information about the history of mathematics.</td>
<td>Explains why mathematics is needed</td>
<td>Specialized Content Knowledge</td>
<td>Create connections to everyday life</td>
</tr>
<tr>
<td>A teacher needs knowledge about the structure of mathematics, because he/she needs to know what prior knowledge the new learning requires.</td>
<td>Know the structure of mathematics</td>
<td>Horizon Content Knowledge</td>
<td></td>
</tr>
<tr>
<td>Self-evidently, subject knowledge plays a major role. You cannot teach mathematics if you do not know mathematics. At school level, there are so many different topics, and a teacher needs to know them and to use that knowledge.</td>
<td>Know mathematical contents to be taught in school</td>
<td>Common Content Knowledge</td>
<td></td>
</tr>
</tbody>
</table>
The data was next converted into Gephi format. *Gephi* software is designed for exploring and manipulating large networks (see Bastian, Heymann, & Jacomy, 2009). A network is a mathematical construction made up of vertices, nodes, or points that are connected with edges, arcs, arrows, or lines. In this study, all of the edges were directed, and therefore we prefer to use the terms nodes and arrows. *Nodes* are knowledge issues mentioned by the student teachers in their essays, while *arrows* mark the ways in which the knowledge issues were connected to each other (see the final column in Table 2). In the case of the network analysis, *out-degree* refers to the number of outward-directed arrows from each node, while *in-degree* refers to the number of inward-directed arrows attached to each node. After the data was imported into the Gephi software, one large network resulted, consisting of 136 nodes and 364 arrows between the nodes. Because the large network is constructed from eighteen student teachers’ ideas, it may be said to represent efficiently the teacher knowledge present in the minds of the future teachers.

To answer the first research question and to examine the sections of the teacher knowledge, attention was paid to the ways in which the different parts of the network were connected to each other. In the context of the network analysis, strongly connected parts of the network are referred to as *communities*. When using the Gephi software, a modularity can be used to detect communities within the network (Blondel, Guillaume, Lambiotte, & Lefebvre, 2008). The modularity value describes whether studied network is arranged in communities of highly connected nodes (Figure 2). *A resolution* in the modularity count is a value used in optimizing the number of communities, which means that a lower resolution is used in order to achieve more communities, while, conversely, a higher resolution can be used to achieve fewer communities. For example, if communities are too large, the resolution can be used for optimizing communities (Fortunato & Barthélemy, 2006). In the present study, a standard resolution of 1.0 was used (Lambiotte, Delvenne & Barahona, 2009). Since the large network
represents teacher knowledge based on student teachers’ ideas, these network communities represent the various sections of the teacher knowledge. Hence, in the present study the term section is used in relation to teacher knowledge, rather than community or modularity. The communities that the data included will be presented in the result section 4.1.

![Figure 2: Left network has three communities and high modularity, and right network has the same number of nodes, edges and communities, but lower modularity (Griffiths, Pedersen, Fenton, & Petchey, 2014)](image)

To find answers to the second research question concerning how the sections of teacher knowledge are connected to each other, we used the Gephi Atlas2 algorithm. The Atlas2 algorithm separates the weakly connected parts, while it keeps the strongly connected parts together (Jacomy, Venturini, Heymann, & Bastian, 2014). Because the modularity count identifies communities and Atlas2 manipulates the layout to show how the various communities are related, it is reasonable to use both of them to explore the ways in which the communities are settled in the network. The network representing the teacher knowledge sections will be presented in the result section 4.2.

We also examined the hierarchy of nodes in the network in order to achieve more detailed answers to the second research question. A partition parameter can be used to identify the properties of nodes or arrows in Gephi that are similar (Khokhar, 2015, pp. 87-96). All of the nodes were classified into the six domains of MKT, and hence this information could be used
in Gephi as a partition parameter. In this case, nodes that are categorized as belonging within
the same domain of MKT will be marked with the same color. Next, a grouping tool\(^1\) was used
to examine the ways in which those six colors (domains of MKT) were connected to each other
(Khokhar, 2015, pp. 209-211). The result will be shown in the result section 4.3.

4 Results

4.1 Teacher knowledge sections

The student teachers’ essays concerned with *The kind of knowledge needed for teaching mathematics* were converted into a single large network. The Gephi software detected eighteen
different teacher knowledge sections (modularity 0.601). Ten of the sections were
interconnected, while eight sections were smaller and separate from the others. The eight
separate sections have only one to four nodes, and hence they were put together and presented
as a single section.

For the sake of clarity, the sections will now be discussed one by one. For this purpose, the
Gephi *Filter Tool* was used to filter out the other sections. Whenever the filter tool was used,
only the internal arrows of the relevant section were examined.

In addition, the nodes have been arranged in the following order. Nodes with no inward directed
arrows (in-degree zero) are on the left in each figure. These issues can be interpreted as knowledge *for something* to occur, or, in other words, they are background knowledge *for doing*
something in the particular section. Nodes that do not have outward-directed arrows (out-degree
zero) are on the right in each figure. These issues can be understood as all requiring background
knowledge, or, in other words, they are a kind of “teaching aims” in the section under
examination. The remaining nodes have both inward- and outward-directed arrows (in- and out-

\(^1\) This option exists in the Gephi 0.8.2. version, which is the one that we used.
degree, one or more in number). These are at the center in each of the Figures. These nodes can be understood as all requiring background knowledge, but at the same time they are also knowledge pertinent to undertaking something in the particular section. An explanation of the sizes and colors of the nodes, arrows, and loops in the Figures is presented in Table 3.

Table 3: Explanation of the sizes and colors of the nodes, arrows, and loops in the Figures.

<table>
<thead>
<tr>
<th>In network</th>
<th>Represents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Node</td>
<td>What kind of knowledge issue were the student teachers discussing? Based on data-based classification.</td>
</tr>
<tr>
<td>Node size</td>
<td>How many arrows penetrate that node? Based on the in-degree value of each node according to Gephi.</td>
</tr>
<tr>
<td>Node color</td>
<td>To which MKT domain is the knowledge issue under discussion related? Based on theory-based classification. This concerns all of the Figures except Figure 12, where the node colors indicate the teacher knowledge sections.</td>
</tr>
<tr>
<td>Arrow</td>
<td>How are the knowledge issues under discussion related to each other? Based on data-based classification.</td>
</tr>
<tr>
<td>Loop</td>
<td>Student teachers mentioned this knowledge issue, but they omitted any understanding of its specific purpose. Based on data-based classification.</td>
</tr>
<tr>
<td>Arrow size</td>
<td>How many student teachers mentioned this relation? Based on edge weight value according to Gephi.</td>
</tr>
<tr>
<td>Arrow color</td>
<td>Which MKT domain does the current arrow indicate? Based on the color of the Gephi target node.</td>
</tr>
</tbody>
</table>

Section 1: Knowing mathematics

The student teachers considered that teachers required knowledge concerned with mathematical theories, terms, rules, concepts, and symbols (Figure 3). As we can see, the issues (nodes) mentioned are on the left in the Figure, which means that they are knowledge for something in the current section. In Figure 3, we are able to see that they comprise knowledge useful for discussing mathematics, advising students during lessons, justifying and proving mathematical theorems, applying mathematics, evaluating how exercises work in practice, and inspiring students. It makes sense that these issues are related as student teachers view them. However,
if the issues are considered in the context of MKT, it can be noted that there are jumps from one domain to another. In this section, the issues on the left are background knowledge and related to *Knowing Mathematics* in general, but they are also connected with issues related to the domains of KCT, CCK, SCK, and KCS.

![Diagram showing connections between different knowledge domains](image)

**Figure 3: Mathematical theories are needed for proving, discussing, and applying mathematics.**

### Section 2: Linking mathematics

The student teachers frequently considered that teachers should create connections between mathematics and other issues (Figure 4). According to the student teachers’ views, teachers connect mathematics with other school subjects, everyday life, and the history of mathematics. As we can see from Figure 4, with this knowledge, teachers can explain why mathematics is needed, answer students’ questions, and promote students’ understanding. On the other hand, promoting students’ understanding is also connected with explaining mathematics in various ways, and furthermore, explaining mathematics is connected with knowing the contents of the mathematics. Conversely, teachers should know mathematics and be able to explain it in many
different ways in order to promote students’ understanding. In this section, many issues are related to creating connections or producing a storyline around mathematics, and hence we have termed this section *linking mathematics*.

![Diagram showing connections between mathematics and everyday life](image)

**Figure 4:** Connections between mathematics and everyday life should be created that will explain why mathematics is needed.

*Section 3: Understanding the structural aspects of learning mathematics*

The student teachers pointed out that teachers need to understand the hierarchy of mathematics and know the structure of mathematics (Figure 5). According to their views, teachers need this kind of knowledge because they need to possess the prior knowledge that is needed for learning a new topic and for examining what students know at that point. Furthermore, they considered that understanding the structure of mathematics and possessing prior knowledge are important, since teachers need to construct new mathematical knowledge on the foundations of the prior knowledge and teachers need to teach how mathematics is constructed. The student teachers also felt that filling the gaps in students’ knowledge requires a knowledge of what the students have learned previously and how mathematics is constructed. All of these issues are related to understanding the *structural aspects of learning mathematics*, which connects almost all of these issues in this section to the definition of HCK.
Section 4: Knowing the common challenges in mathematics

The student teachers considered that teachers need to help their students to overcome difficulties in mathematics, present alternative solutions, and choose suitable exercises (Figure 6). To achieve these aims, teachers require background knowledge that will enable them to recognize students’ mathematical ideas or the inaccuracies in their reasoning. In this context, the student teachers mentioned the pitfalls of mathematics that hinder learning, rather than learning difficulties in general. In this particular section, the student teachers most frequently discussed knowing the general challenges in mathematics, which in this case are related to SCK. However, the issues in this section are also related, for example, to organizing remedial education, which in turn connects SCK to KCT.
Figure 6: An awareness of the general challenges posed by mathematics is necessary for the teacher to help students in their learning of mathematics.

Section 5: Knowing different teaching methods

The student teachers’ considered that teachers need knowledge in order to be able to present suitable questions and to provide feedback, because teachers need to control the interaction in the classroom. As can be seen from Figure 7, controlling interactions happens both between students and also between students and teachers. According to the student teachers’ views, avoidance of teacher-centered teaching methods or changing a seating plan can help in controlling the interaction between students and the teacher.

Knowledge about different teaching styles is also related to motivating students, considering the dissimilarities between students, and promoting students’ interest in mathematics. According to the student teachers’ views, knowing how mathematics is applied, creating mathematical exercises, and recognizing the difficulty-level of exercises are also related to motivating students and promote their interest in mathematics. According to the student teachers’ perceptions, knowing teaching methods seems to be “an intersection node” in this section, because knowing different teaching methods can help in controlling interaction that is
related to KCT. On the other hand, knowing teaching methods can also help to motivate students in considering the dissimilarities between students in issues related to KCS.

![Diagram](image)

**Figure 7: Different teaching methods are needed in order to motivate students and to control interaction.**

**Section 6: Choosing appropriate teaching methods**

Various issues are closely connected with the choice of appropriate teaching methods since node sizes express how many arrows are actually penetrating the topic (the in-degree of the node). As can be seen from Figure 8, student teachers felt that underlying the choice of appropriate teaching methods is background knowledge related to KCS and SCK.

According to the student teachers, teachers require theoretical knowledge about learning that will enable them to recognize the best ways to learn and also the range of available learning methods before they can choose the most appropriate teaching methods for a particular teaching situation. In addition, the student teachers considered that teachers need to know their student
groups, to identify their students’ learning styles, and to consider the dissimilarities between student groups before they can choose appropriate teaching methods. These background knowledge issues are related to the domain of KCS, whereas the selection of teaching methods is also related to KCT.

Conversely, the student teachers felt that, before an appropriate teaching method can be selected, teachers need knowledge in order to be able to choose the most relevant ways of presenting mathematics and also approaching mathematical content. According to our classification, these issues are most closely related to SCK, since in this case the student teachers were talking about the special nature of mathematics. As we can see from Figure 8, choosing appropriate teaching methods seems to require background knowledge related to the domains of KCS and SCK, but, by making an appropriate selection, teachers can promote students’ group work and interaction skills.

![Diagram of Theoretical and practical knowledge related to learning](image)

**Figure 8: Theoretical and practical knowledge related to learning are prerequisites for choosing an appropriate teaching method for each teaching situation.**
Section 7: Promoting students’ different ways of learning

In this section, most issues are connected to promoting students’ learning and taking students’ dissimilarities into account (see node sizes in Figure 9). These issues were classified under KCS since in this particular case the student teachers discussed their evaluation and understanding of students’ behavior.

According to the student teachers, supporting the dissimilarities between students’ ways of learning in their own teaching requires taking students’ learning styles into account, choosing appropriate teaching materials, presenting mathematics in various ways, examining mathematics from various perspectives, using different examples, and manipulating the mathematical contents. However, as Figure 9 illustrates, presenting mathematics in various ways and examining mathematics from various perspectives requires a deep understanding of the contents of mathematics. Generally speaking, the same issues are connected to promoting students’ learning. According to the student teachers, choosing an effective teaching sequence, getting the students to participate, and designing a good lesson plan are also linked to promoting students’ learning. In this section, most issues concerned promoting students’ different ways of learning. The colors of the arrows indicate that most issues are directed toward the domain KCS (mostly red arrows).
Figure 9: Developmental and mathematical theories are needed to support students’ dissimilarities and to promote their learning.

Section 8: Separating teaching

The student teachers considered that, in addition to teachers needing to know their students, they also need to recognize the kind of general challenges that students may face in learning mathematics, and also their learning difficulties and potential problems in terms of reading skills and concentration (see Figure 10). In this case, the student teachers devoted some attention to the general difficulties faced in learning mathematics. According to the student teachers, this kind of knowledge is yet another form of background knowledge for differentiating the teaching and organizing learning support for students. In this section, the student teachers’ observations related to students can be identified as KCS, although actions taken with regard to Separating teaching and organizing learning support are related to KCT. Hence, this section strongly connects the KCS domain with that of KCT.
Figure 10: Identifying challenges in learning are necessary for differentiating modes of teaching.

Section 9: Assessing and improving teaching

According to the student teachers’ views, mathematics teachers should be able to evaluate their own teaching and also to evaluate their own teaching based on student feedback (see Figure 11). Furthermore, teachers need to be aware of educational trends and possess research-based knowledge about learning mathematics. All of these issues are directed toward improving teachers’ own teaching in the current section. Hence, we term this section Assessing and improving teaching, which relates in the main to KCT.

Figure 11: Self-evaluation skills are necessary for improving teaching.
Section 10: Using technology

The student teachers were of the opinion that choosing appropriate technological aids, software, and other learning aids would require background knowledge about technological development (Figure 12). By technological development, student teachers meant that teachers should be aware of the most recent technological equipment used in teaching. In addition, a teacher also needs to master the use of technological equipment such as calculators, software, applications, and other concrete learning aids. According to the student teachers, choosing appropriate teaching aids can make teaching interesting and diverse. In addition, the student teachers felt that technology can be also used to illustrate the special characteristics of mathematics. This section relates to Using technology, although some of the issues in this section connect the KCC domain to those of KCT and SCK.

Figure 12: Technological aids are necessary for illustrating mathematics.

Section 11: Following the guidelines of the national curriculum

The student teachers considered that a teacher needs to know the contents of their national curriculum in order to be able to follow its guidelines, to know what they should teach, and to
enable their students to study according to the stipulations of the curriculum (Figure 13). The student teachers also regarded it as necessary for teachers to search for new exercises, e.g. on the internet, and to collect their own bank of teaching-related materials, which in turn would help them to select appropriate exercises for their teaching. Knowing pedagogical concepts helps in the discussion of teaching with colleagues, and displaying an interest in mathematics can make a significant difference in promoting the individual teacher’s own mathematical proficiency. Based on the thickness of the arrows (i.e., the number of student teachers), this section relates most closely to **Following the guidelines of the national curriculum.**

![Diagram](image)

**Figure 13: Knowing the contents of the national curriculum is a prerequisite for following its guidelines.**

### 4.2 Teacher knowledge sections within the main network

Figure 14 represents how the teacher knowledge sections are related to each other. Because the Gephi Atlas2 algorithm software separates the weakly connected parts and keeps strongly connected parts together, sections that have more likely connections between each other are
considered to be “neighbours”. Figure 14 shows, for example, that the section Knowing mathematics is related to the section Linking mathematics. This obviously makes sense since teachers first need to know mathematics and then they can link mathematics to other areas. Knowing the common challenges in mathematics also bears a relationship, for example, to Separating teaching and Choosing appropriate teaching methods.

Figure 14: This figure represents how the various teacher knowledge sections (=colors) are related to each other. The layout was produced with the aid of the Atlas2 algorithm. Interestingly almost all the sections are linked to Choosing appropriate teaching methods and Promoting students’ different ways of learning (see Figure 14). These two sections, which are strongly related to each other, form the center of the network. From this perspective, the
knowledge required for teaching mathematics appears to find its focus in two questions in student teachers’ minds: “How can appropriate teacher methods be selected?” and “How can different ways of learning be promoted?”. Interestingly, Following the guidelines of national curriculum has no links to the other sections. This suggests that the student teachers expressed issues related to the contents of the national curriculum and other issues in this section as separate entities.

4.3 The structure of teacher knowledge

Because all of the 136 nodes were classified into six domains of MKT, the Gephi Grouping tool was used to investigate how the nodes were connected to each other. All of the internal and external arrows of all the sections were taken into account to explore how the six domains of MKT were related to each other in the minds of future teachers. The result is presented in Figure 15. The node sizes were based on the in-degree value of the specific node.
Figure 15: The result demonstrates that the six domains of MKT are in hierarchical order.

Figure 15 reveals that Common Content Knowledge has an out-degree total of 30 and an in-degree total of zero. This suggests that all of the issues related to Common Content Knowledge are background knowledge related in the minds of student teachers to doing something. Similarly, Knowledge of Content and Curriculum has an out-degree total of 37 and an in-degree total of only three. This indicates that Knowledge of Content and Curriculum is also background knowledge related to doing something. According to the results, both CCK and
KCC represent background knowledge for the other knowledge domains, and hence we identified them as *Foundation knowledge* for teachers.

The in-degree and out-degree totals are in balance for the domains of Horizon Content Knowledge and Specialized Content Knowledge because the same numbers of arrows are directed inside and outside in case of these two domains. The result presented in section 4.1. shows that student teachers think that teachers need a background knowledge of mathematical theories in order to be able to draw up connections between mathematics and everyday life, other school subjects, and the history of mathematics. In general, these student teachers seem to think that teachers convert the content of their subject into a form that will help students understand why and where mathematics is needed. In this phase, teachers transform subject knowledge from its form as it exists in Common Content Knowledge into a form of Specialized Content Knowledge. This transformation requires both foundation knowledge and also knowledge concerning the special characteristics of mathematics. Similarly, the student teachers thought that a teacher needs to master the various uses of technology and other teaching aids, which can therefore be classified as familiarity with Knowledge of Contents and Curriculum. The students’ teachers thought, however, that teaching aids, especially technology, could be used, for example, to illustrate mathematics. The same transformation happens in this phase. The process begins with mastering teaching aids and continues to the selection of suitable ways to use the aids. When teachers attempt to illustrate mathematics knowledge itself, knowledge of the particular characteristics of mathematics is a prerequisite. According to the results, the domains of Horizon Content Knowledge and Specialized Content Knowledge are related to changing the form of the knowledge, and hence those domains are termed *Transformation knowledge* for teachers.

According to the results represented in Figure 15, the domains of Knowledge of Content and Students and Knowledge of Content and Teaching are the final phase in the totality of teacher
knowledge. Both types of Foundation and Transformation knowledge have been consigned to these two domains. The domains of Knowledge of Content and Students and Knowledge of Content and Teaching are also strongly connected to each other. This makes sense, because teachers need to know learning theories and teachers also make empirical observations about their students. According to the student teachers’ views, both of these are effective starting points for the selection of appropriate teaching methods. Because teachers use their knowing of how students learn (in theory and in practice) in order to make decisions about how to teach, the domains of Knowledge of Content and Students and Knowledge of Content and Teaching are linked together. Teachers may recognize the challenges inherent in student learning or they may note that students struggle in learning mathematics (in this case, Specialized Content Knowledge will also be involved), which has an effect on the kind of decisions that the teacher will make in those situations. According to the results, decisions that include the selection of appropriate teacher methods and of promoting students’ different ways of learning are related to almost all of the sections, and hence these two questions, “How should appropriate teaching methods be selected?” and “How can different learning be supported?” also activate all of the domains of MKT. According to the results, the domains of Knowledge of Content and Students and Knowledge of Content and Teaching are best related to actual teaching events, and hence we have termed the domain Operation knowledge for teachers.

These results show that, at least in the minds of the student teachers investigated here, there exists an MKT hierarchy. Common Content Knowledge and Knowledge of Contents and Curriculum are, above all, Foundation knowledge (connections mostly outside), while the domains of Horizon Content Knowledge and Specialized Content Knowledge are primarily Transformation knowledge (in- and out-connections in balance) and the domains of Knowledge of Content and Students and Knowledge of Content and Teaching are principally Operation knowledge (connections mostly inside) for teachers.
5 Discussion

These findings have encouraged us to argue that in studying teachers’ perceptions of what kind of knowledge is needed for teaching it is just as important to study how teachers regard that knowledge. However, when the relationships that exist in the minds of teachers are considered, the research becomes somewhat more complex. Regardless of this complexity, however, we have demonstrated that network analysis can offer a number of useful tools for examining relationships of this kind.

The structural features of teacher knowledge

The most interesting result was the finding that the six MKT domains exist in hierarchical order in the minds of future teachers. Based on this finding, the student teachers were of the opinion that Common Content Knowledge and Knowledge of Contents and Curriculum are more like background knowledge and were needed for doing something. This means that these two domains of MKT are background knowledge for the other four domains of MKT.

Because of the hierarchy, the result indicates that to understand Specialized Content Knowledge and Horizon Content Knowledge, a teacher should first acquire Common Content Knowledge and Knowledge of Contents and Curriculum. This makes sense, because to understand the structure of mathematics (HCK) teachers should have a broad knowledge of several mathematical contents (CCK). Teachers also need to know how the contents are organized in the national curriculum (KCC) in order to understand the most effective sequence in which mathematics can be presented to students (HCK). In addition, teachers require a deep understanding of the different mathematical theories (CCK) in order to be able to present mathematics, to demonstrate relevant examples, to create connections, and to apply the mathematics (SCK).
According to the results, all four domains (CCK, KCC, SCK, and HCK) are related in different ways to the last two domains. Thus, these last two domains (KCS and KCT) will require the most background knowledge. It is possible, therefore, that this is why the student teachers may consider that understanding Knowledge of Content and Teaching and Knowledge of Content and Students is the most challenging. Our findings support this interpretation because the issues related to Knowledge of Content and Students and Knowledge of Content and Teaching are connected in many different ways to the other four MKT domains. It is, however, possible that this kind of result could be used to develop future teachers’ thinking, for example, in the following way. Our results indicate that questions such as “How can an appropriate teaching method be selected? and “How can different kinds of learning be supported?” are related simultaneously to several sections of teachers’ knowledge. Hence, asking student teachers to pondering on the possible answers to such questions may well be an effective assignment to activate and develop future teachers’ thinking at a developed level.

In general, of course, the way in which student teachers view teacher knowledge may not be perfect. Nevertheless, it may be useful for researchers to consider generally the notion that some parts of teacher knowledge are background knowledge and are necessary for undertaking some activities. This, in turn, points toward the phenomenon that teacher knowledge includes sequences, meaning that one part of teacher knowledge is generally situated between others. The issues situated in the middle of these sequences may indeed be critical to the process of learning to teach mathematics. In other words, to gain an understanding of the so-called middle parts of teacher knowledge, teachers need background knowledge, and to gain an understanding of the totality of teacher knowledge, teachers require an understanding of the middle parts of teacher knowledge. Or, to re-state the same perception differently: if teachers do not understand the middle parts of teacher knowledge, this may cause serious problems for their understanding of how the domains of teacher knowledge are interrelated.
Our findings indicate that Specialized Content Knowledge and Horizon Content Knowledge may be good candidates for the role of “middle part” or “critical” teacher knowledge. If we omit these two domains in teacher education, the necessary totality of teacher knowledge may very well collapse. Our findings support this interpretation, as we can see from Figure 15. If we remove Transformation knowledge (HCK and SCK), then most of the links between Foundation knowledge (CCK and KCC) and Operation knowledge (KCS and KCT) disappear. This could mean that teachers find that they have learned mathematics and pedagogy, but they cannot connect those two knowledge types properly in their minds. When teachers feel that they cannot connect mathematics and pedagogy, some of the benefits of teacher knowledge will undoubtedly disappear. In broad terms, therefore, it can be said that our finding means that knowing Common Content Knowledge is insufficient for teaching, since that knowledge should be first transformed into Specialized Content Knowledge until the mathematical knowledge becomes activated in different teaching situations.

This finding also raises the question of how the contents of mathematics teacher education should be scheduled. Should teachers first acquire a knowledge of mathematics and then a knowledge of pedagogy to understand what effective teaching is about? Some studies support this approach to scheduling the contents in mathematics teacher education (e.g. O’Meara, 2011). Another way to schedule the contents of teacher education is, however, to mix the contents of subject and pedagogy, as in the spiral curriculum (Harden & Stamper, 1999). This is the approach that is preferred in Finland. In the case of Finland, the spiral curriculum means that student teachers acquire some knowledge of mathematics and of pedagogy and then they practice their own teaching. Student teachers may, for instance, note that they are not yet professionals and that they need further knowledge and skills to become competent teachers. This kind of realization is an important factor in reflecting their own learning and in increasing their motivation to continue their courses and deepen their understanding of mathematics,
pedagogy, and teaching practice. Sometimes, more difficult learning can lead to better long-term learning results (Schmidt & Bjork, 1992). Since learning resembles a spiral, this approach may help teachers to acquire a model for developing their professional skills even after graduation. Whatever the truth is, scheduling may well require more research attention in the context of mathematics teacher education.

**Using the method for evaluating and improving the contents of mathematics teacher education**

Because our findings have been based on a description of described how student teachers viewed the kinds of knowledge required for teaching mathematics, it may not be a “perfect” view. If the picture is not perfect and links or sections are missing, new components can be developed within the framework of mathematics teacher education in order to fix the problem. Hence, it is our opinion that our findings can be used to develop the contents of mathematics teacher education by fixing “missing” links and supporting “existing” links. This kind of strategy, where the information gleaned from evaluation is used to develop the educational setting, follows the principles of Design-Based Research (Design-Based Research Collective, 2003; Edelson, 2002).

In support of existing links, the student teachers thought that teachers should know, for example, how to use teaching equipment to visualize mathematics (which connects KCC and SCK) or to create connections between the mathematical contents and everyday life (which connects CCK and SCK). Because links of this kind already exist in the minds of student teachers, there would be a further match with student teachers’ ideas if teacher educators were to show how different types of technology can be used in great variety of ways to illustrate different mathematical phenomena or to demonstrate how mathematical contents are really connected to different everyday applications and life in general. If teacher educators were to
corroborate student teachers’ views, it would also be reasonable to say that this might well support student teachers’ learning to better effect.

On the other hand, in order to fix missing links student teachers suggested that knowledge concerned with *Following the guidelines of the national curriculum* was not connected to the other sections of teacher knowledge. In other words, this finding may indicate that the teacher education program may form too few connections between the contents of the national curriculum and the teaching of mathematics. This may, in turn create problems since the national curriculum consists of a formulation of the general aims of the teaching at each level of school throughout the education system.

In the course of our research, we explored student teachers’ perceptions of the knowledge required for teaching mathematics, and we used MKT as a framework in support of their perceptions during the final year of their education. The effects of their teacher education were expected to be visible in their views, since they had completed almost all of their mathematical and pedagogical courses and had finished almost all of their teaching practice. Hence, the results of the study provide a means of reflecting the impact of their teacher education. If we now summarize our results as a whole, the results suggest that the future teachers thought that mathematics teacher education should include particular contents aimed at

- knowing and linking mathematics (Sections 1 and 2)
- knowing the general challenges and structural features of mathematics (Sections 3 and 4)
- knowing and choosing teaching methods (Sections 5 and 6)
- promoting students’ different ways of learning and separating teaching (Sections 7 and 8)
- assessing and improving teaching (Section 9)
- following the guidelines of the national curriculum and making use of the available technology (Sections 10 and 11)
Our findings raise the following question: Should the contents of mathematics teacher education be based on a theoretical framework from the very beginning? It is clear that different teacher education programs affect a teacher’s knowledge and skills in different ways (Darling-Hammond, Chung & Frelow, 2002; Darling-Hammond, Holtzman, Gatlin, & Heilig, 2005). Teacher knowledge has, however, an impact on student achievement (Hill, et al., 2005), and research has shown that, eventually, the contents of mathematics teacher education will also have an influence on students’ learning (Monk, 1994). In addition, the contents of mathematics teacher education have an impact on future teachers’ knowledge, and it can even be claimed that future teachers’ knowledge at least partly explains why some countries enjoy greater success in international evaluations, such as TIMSS (Schmidt, Houang, & Cogan, 2011). Teacher knowledge is commonly used as a single part of the conceptual framework in assessing the success of teacher education programs (e.g. TEDS-M²), but it seems logical that the contents of teacher education could also be placed within the same framework. If the contents of teacher education are examined from the perspective of teacher knowledge, this may help us to see which domains of teacher knowledge are actually covered. Locating the contents within a review of teacher knowledge may be one way of developing an improved curriculum for mathematics teacher education.

The future of teacher knowledge research

A network analysis may then be a useful way to conduct future research into teacher knowledge. Nowadays, software can handle massive amounts of data, and there is usually an option to add new data to the old. On the other hand, with a single click the old data can also, if necessary, be filtered out. Hence, the method presented in the present study can be used for comparing

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2 Teacher knowledge plays a major role in the conceptual framework of *Teacher Education and Development Study in Mathematics* (TEDS-M), where approximately 750 mathematics teacher education programs in seventeen countries were evaluated (e.g. Tatto, et al., 2008).
results from a variety of different countries and target groups. Comparison of experts’ and
novices’ views on the knowledge needed for teaching might then provide further information
about the knowledge-growing process involved in the transformation of novices into experts.
On the other hand, looking at the differences between in-service and pre-service teachers’ views
may well provide information about the knowledge-growing process that occurs during teacher
education itself. Furthermore, combining teacher educators’ and school teachers’ perceptions
may also reveal some of the contradictions and points of consensus that could help in the
development of school teaching and teacher education.

The next stage for this study would be the construction of a large international database that
could be used to construct a new model of teacher knowledge. A large database of this kind
could be used to examine differences but also to develop a new consensus model of teacher
knowledge that will take account of how knowledge is related in the minds of respondents. A
new direction of this kind in research into teacher knowledge might, in sum, help us to deepen
our research-based understanding of teacher knowledge.

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This dissertation presents new methods of applying a Mathematical Knowledge for Teaching (MKT) framework in the context of mathematics teacher education. MKT can be used as a tool for examining developmental needs in mathematics teacher education. The present study also demonstrates an innovative way of using MKT and network analysis as tools in the investigation of teacher knowledge.